ESTABLISHING AND MAINTAINING DATABASES OF SELF-AFFINE TILES

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Numeration 2018 – Paris, 23 May

1. Basic concepts
2. Search – demonstration
3. Neighbor maps – main tool
4. Some results

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All credit for programming goes to Dmitry Mekhontsev!

1.1 BASICS

$\mathbb{R}^d$, or $\mathbb{C}$ for $d=2$

Focus on Mappings and Equations ("IFS")
(appropriate data for computer)

Numeration system

$$\underset{\uparrow\,\text{expansive map}}{g}(A) = \underbrace{h_1(A) \cup \ldots \cup h_m(A)}_{\substack{\text{digits} \\ \text{(isometries)}}}$$

Decimal system: $A=[0,1]$, $g(x) = 10x$,
$h_k(x) = x+k$, $k = 0,1,\ldots 9$.

1.2 Expansion

$$g(x) = M \cdot x$$

Examples: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, algebraic numbers, $|\lambda_1|, |\lambda_2| > 1$.

$g(z) = \gamma \cdot z$, $|\gamma| > 1$ complex algebraic number

General case: M rational matrix on high-dim. $\mathbb{R}^d$
and we consider 2-dim. eigenspace with $|\lambda| > 1$.

Typically, M is the companion matrix of a polynomial with integer coefficients.

Note: Rational arithmetics makes calculations accurate!

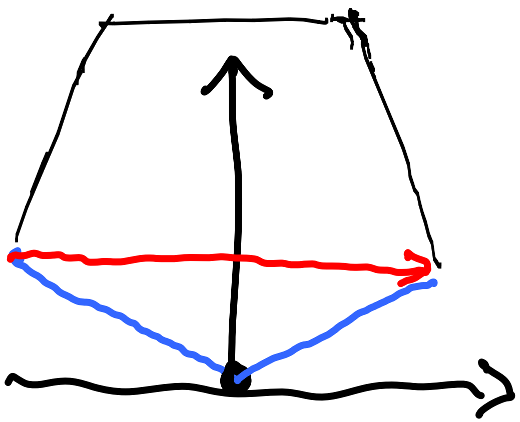
1.3 Digits

$h(x) = s(x+v)$ v integer vector

$s \in S$ – symmetry group which commutes with the expansion g (rotations and reflections in eigenplane)

Problem. Which g and S fit together to give tiles?

Example (Penrose–Robinson)



$s(z) = e^{i\frac{\pi}{5}} z$ (36° rotation)

$g(z) = \tau \cdot z$ $\tau = \frac{\sqrt{5}-1}{2}$

$g = s - s^4$

$r(z) = \bar{z}$ $S = \langle s, r \rangle$

1.4 Equations

$$g(A) = \bigcup h_k(A)$$

$$A = \bigcup g^{-1} h_k(A) = \bigcup \underbrace{f_k}_{\text{contraction}}(A)$$

Hutchinson, "IFS"

$$A = g^{-1}h_1(A) \cup g^{-2}h_2(A) \cup g^{-3}h_3(A)$$ Rauzy

Most general: graph-directed systems

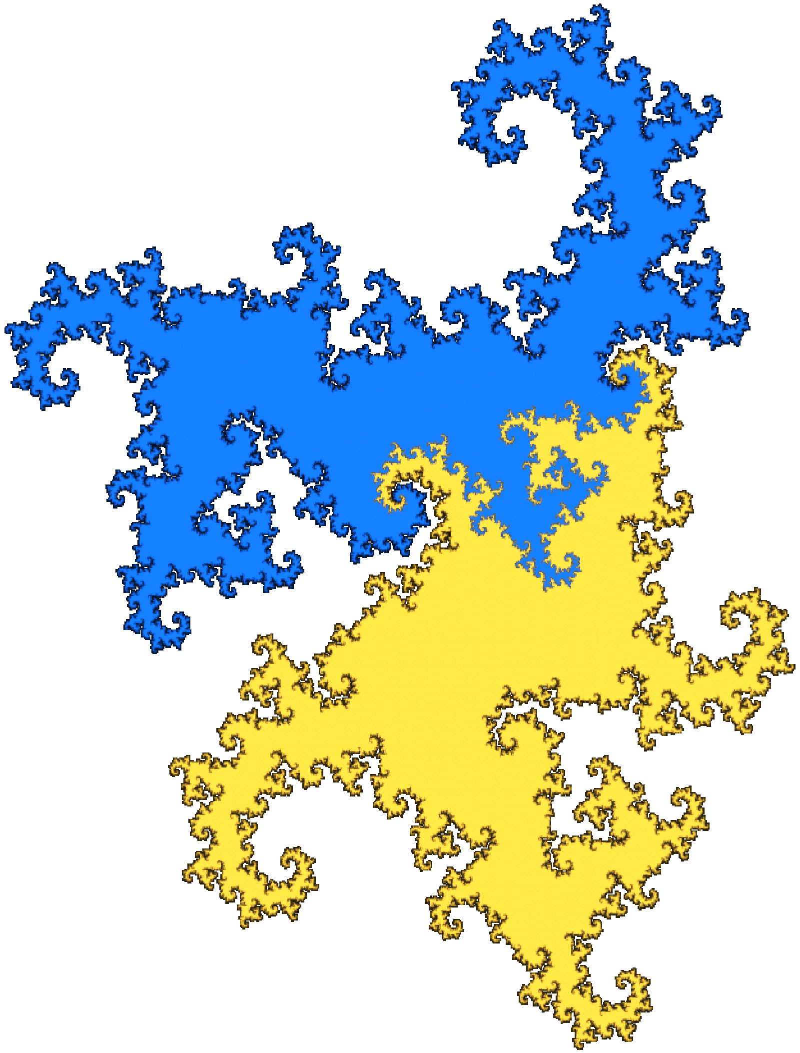
$$C_i = \bigcup_{(j,k)\in Q_i} f_k(C_j) \qquad i = 1, 2, \ldots, n$$

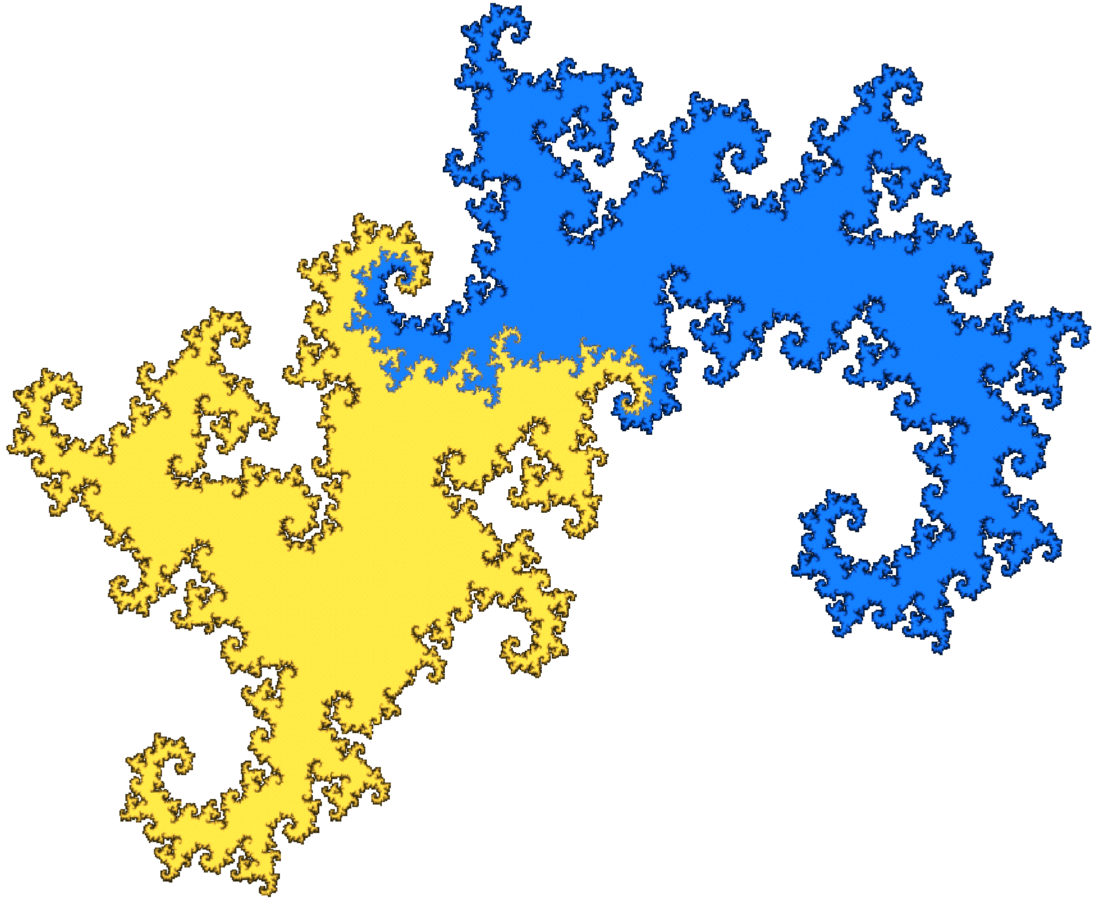
Mauldin + Williams, Thurston, Kenyon, ...

1.5 Example

$g(A) = h_0(A) \cup h_1(B)$, $g(B) = h_2(A) \cup h_3(B)$

$g(z) = (1+i)z$, $h_0 = -z$, $h_1 = i\bar{z} + i$, $h_2 = \bar{z} + 2$, $h_3 = z+1$





2.1 SEARCH CONCEPT

FAMILY given by

g, S, equations

EXAMPLES specified by

digits $h_1 \dots h_m$

$h(x) = s(x+v)$, $(s,v) \in S \times \mathbb{R}^d$

Random walk on $(S \times \mathbb{R}^d)^m$

to find good examples

```
@@version 3

@G
$n=FAMILY
$dim=2
g=[1,-1,1,1]
s=[0,-1,1,0]
r=[-1,0,0,1]
&R=$semigroup([s,r])
&T=R*$vector(0)
h0=R
h1=T
h2=T
h3=T
A=g^-1*(h0*A|h1*B)
B=g^-1*(h2*A|h3*B)

@:G
$n=EXAMPLE 1
h0=-1
h1=-1*[0,-1]
h2=1
h3=s*[0,1]
```

$g = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$h_0 = x + v$

$h_k = r(x+v)$

equations

$h_0 = -x$

$h_1 = -\left(x + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)$

$h_2 = x$

$h_3 = s\left(x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$

2.2 Tiling condition for family (necessary)

Obtained from comparing areas in the equations

- $\det g = m$ for $g(A) = \bigcup_{k=1}^{m} h_k(A)$

- $t^1 + t^2 + t^3 = 1$ with $t = |\lambda|^{-2}$

  for $A = g^{-1} h_1(A) \cup g^{-2} h_2(A) \cup g^{-3} h_3(A)$

- $|c|^2 = 2$ for $g(z) = cz$, $g(A) = h_0 A \cup h_1 B$, $g(B) = h_2 A \cup h_3 B$

- The equations determine a characteristic polynomial for g.

2.3 Search procedure

Random walk on $(S \times \mathbb{R}^d)^m$
to find good examples

For each new parameter

- check whether example yields a tiling

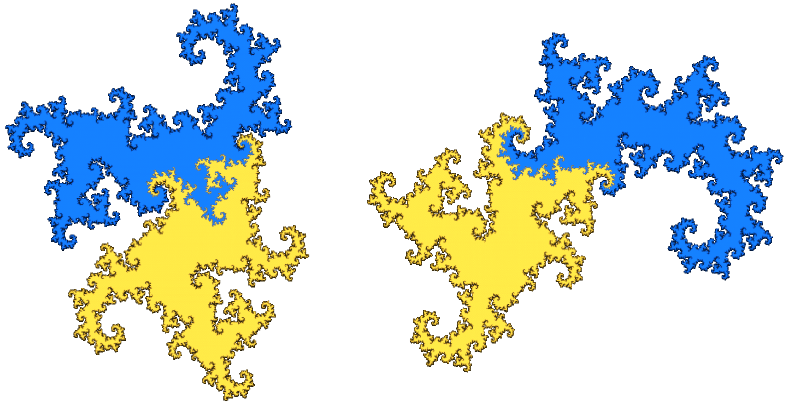
if yes

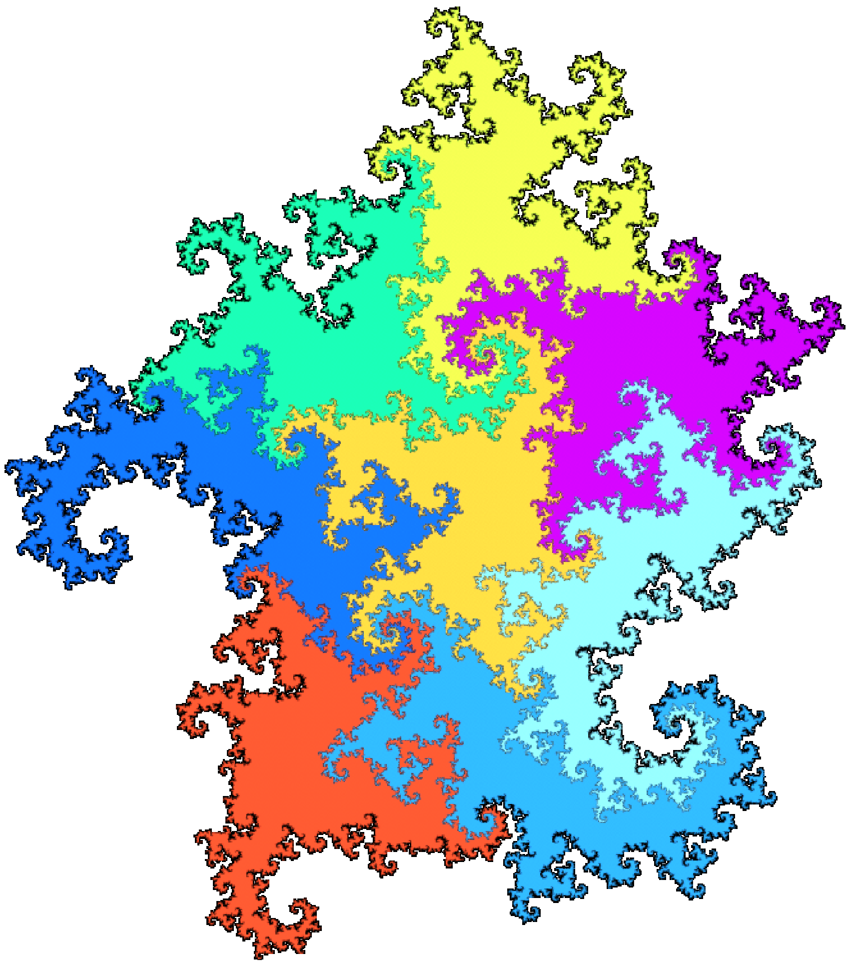
- check whether the example is NEW
  (not isomorphic to a previous example) difficult

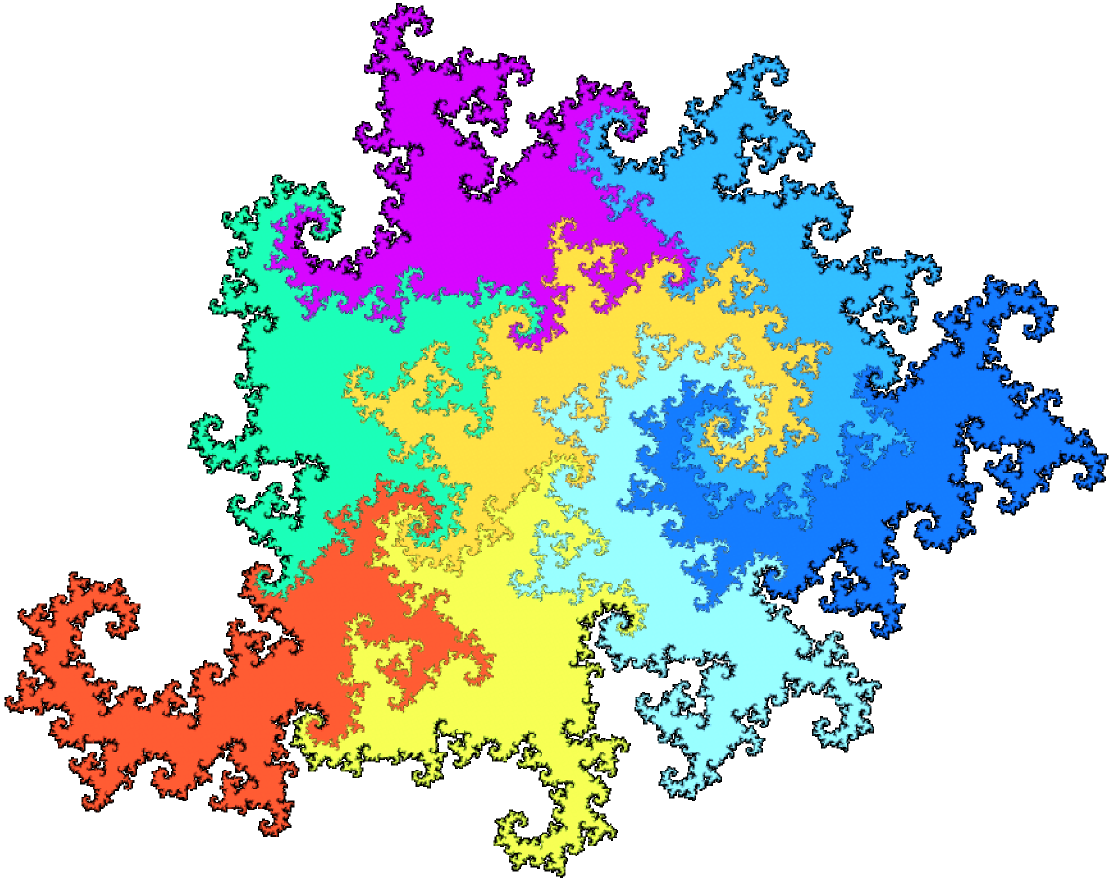
if yes

- add the example to the list and
  determine its properties needed to compare with next examples

3.1 Neighbors in Tilings



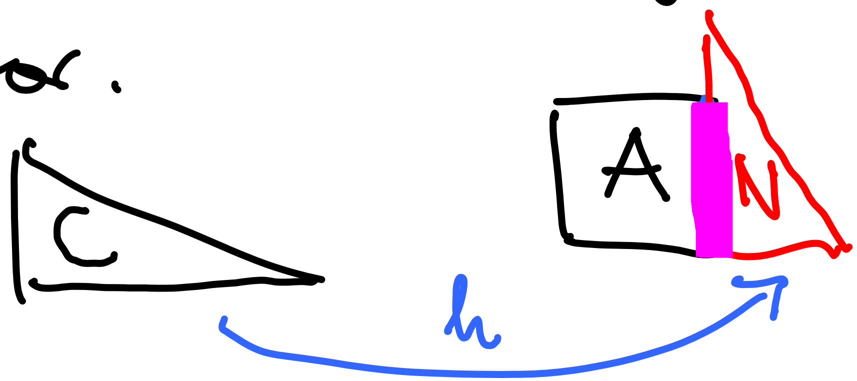




3.2 Neighbor maps and boundary sets

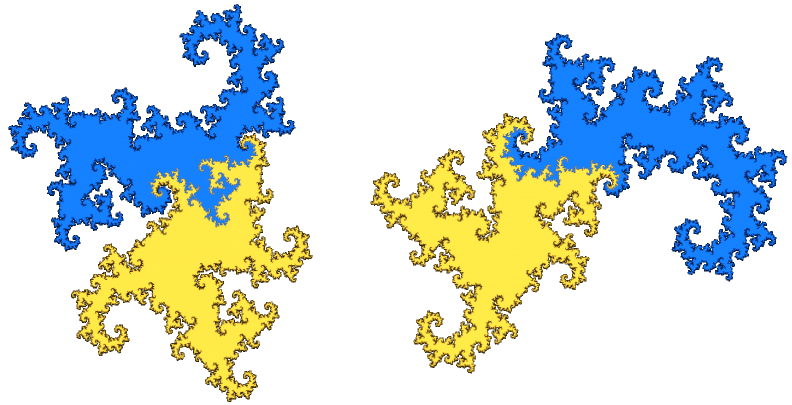
Well-known technique, papers by Akiyama, Bandt, Berthe, Loridant, Ngai, Rao, Siegel, Steiner, Thuswaldner, ...

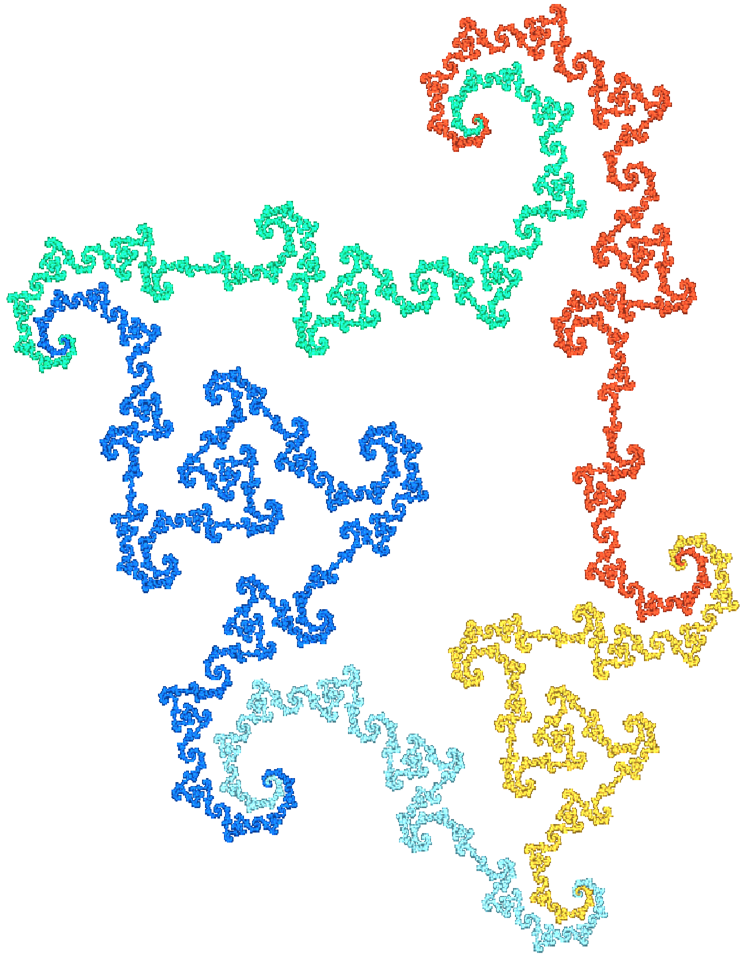
To every potential neighbor N of A in a tiling there is a map h with $h(C) = N$ where C is the type (= prototile) of the neighbor.

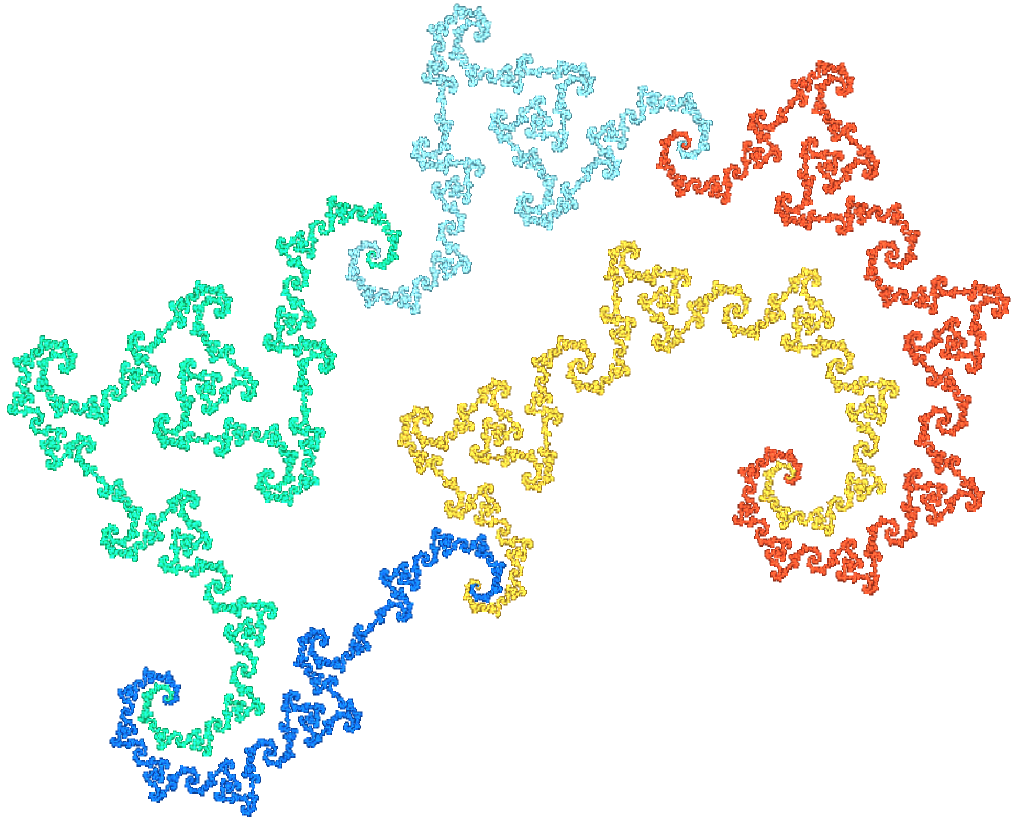


Moreover, there is a corresponding boundary set $B = A \cap N$ of A.

3.3 Boundary sets for our example





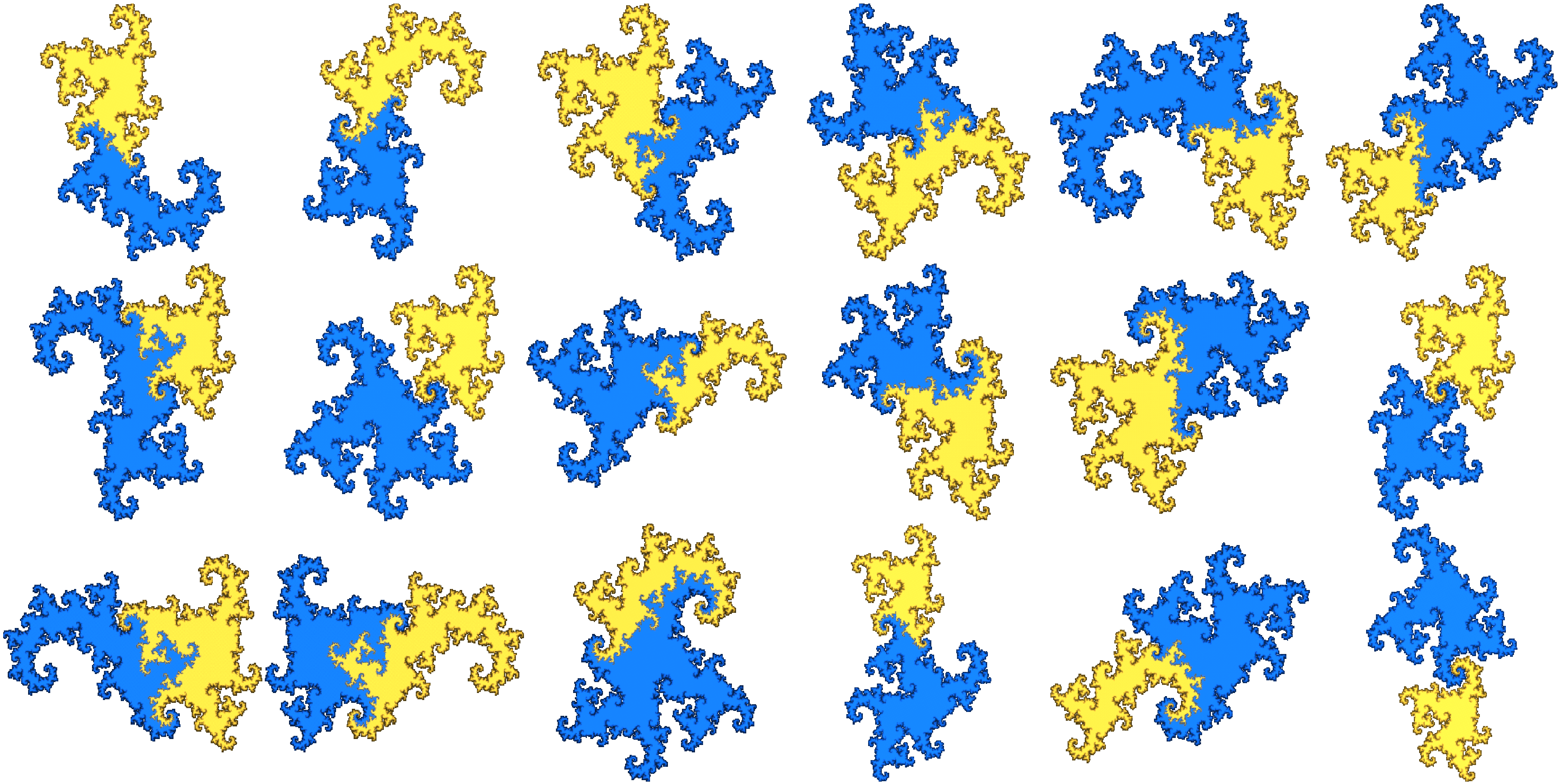


3.4 The neighbor algorithm

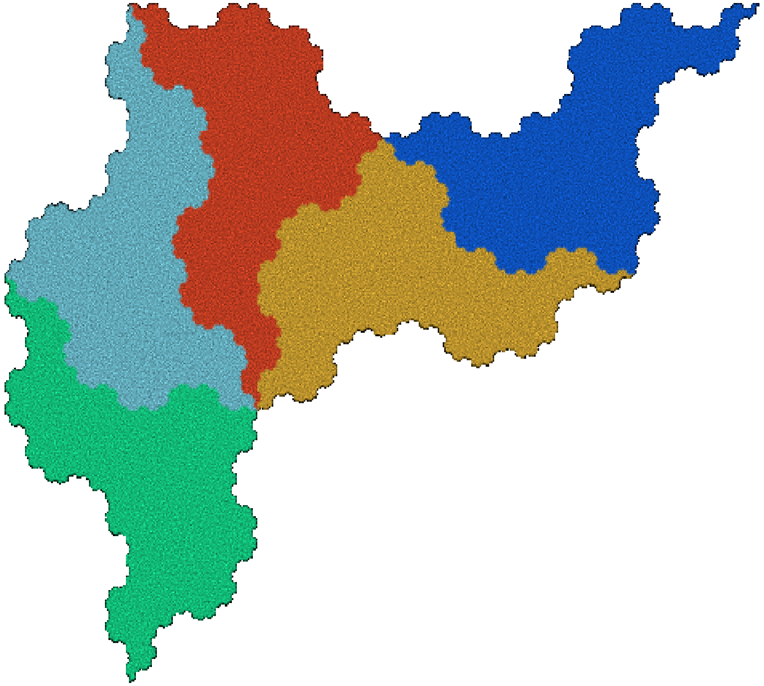
- Neighbor maps can be determined in a recursive way. If there are only finitely many neighbor maps, and the identity map is not a neighbor map, then we have a finite type tiling.
- The search algorithm selects only finite type tilings with $\leq N$ neighbor maps, for $N = 200$, say.
- The neighbor maps are arranged as a graph from which the dimension of the boundary and various topological properties of the tiles can be determined.

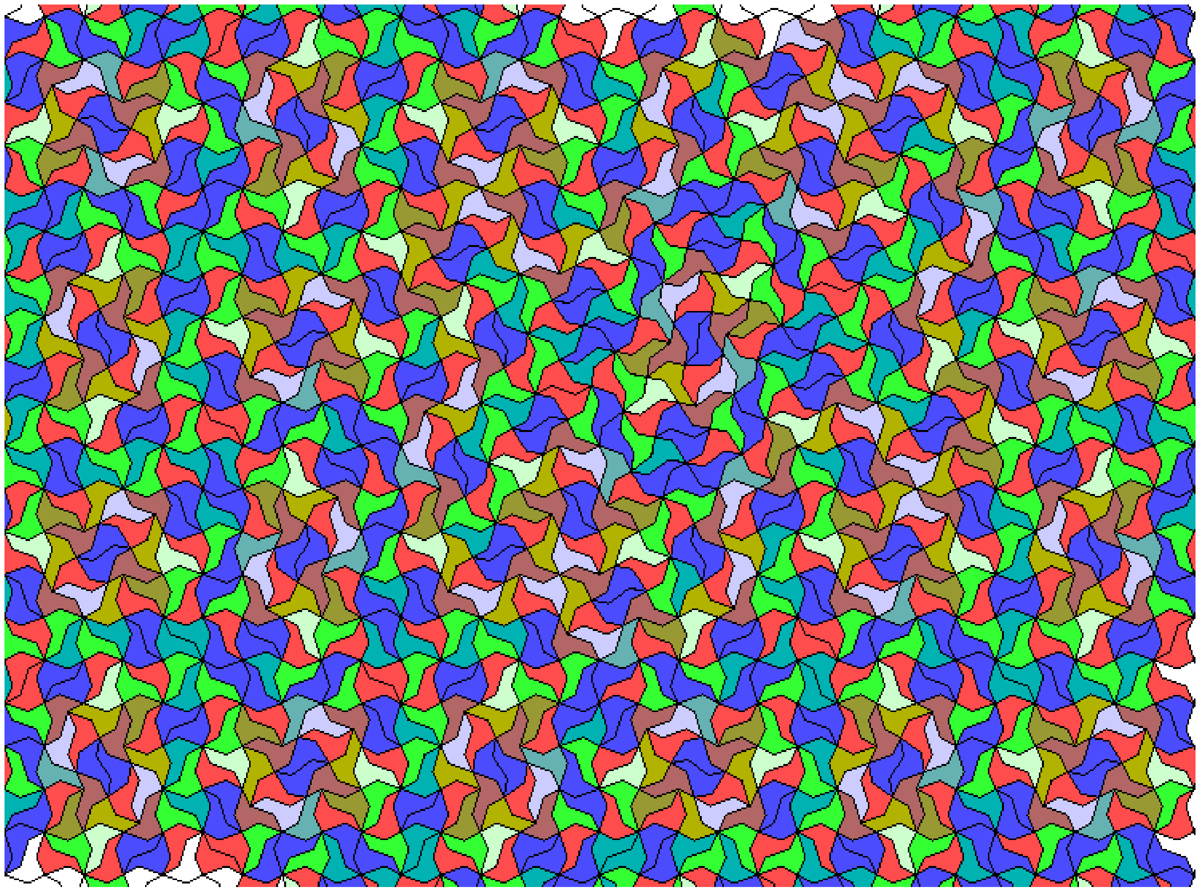
Some neighbors for our example

yellow : basic tile
blue : neighbor



4.1 Example: the fractal pinwheel

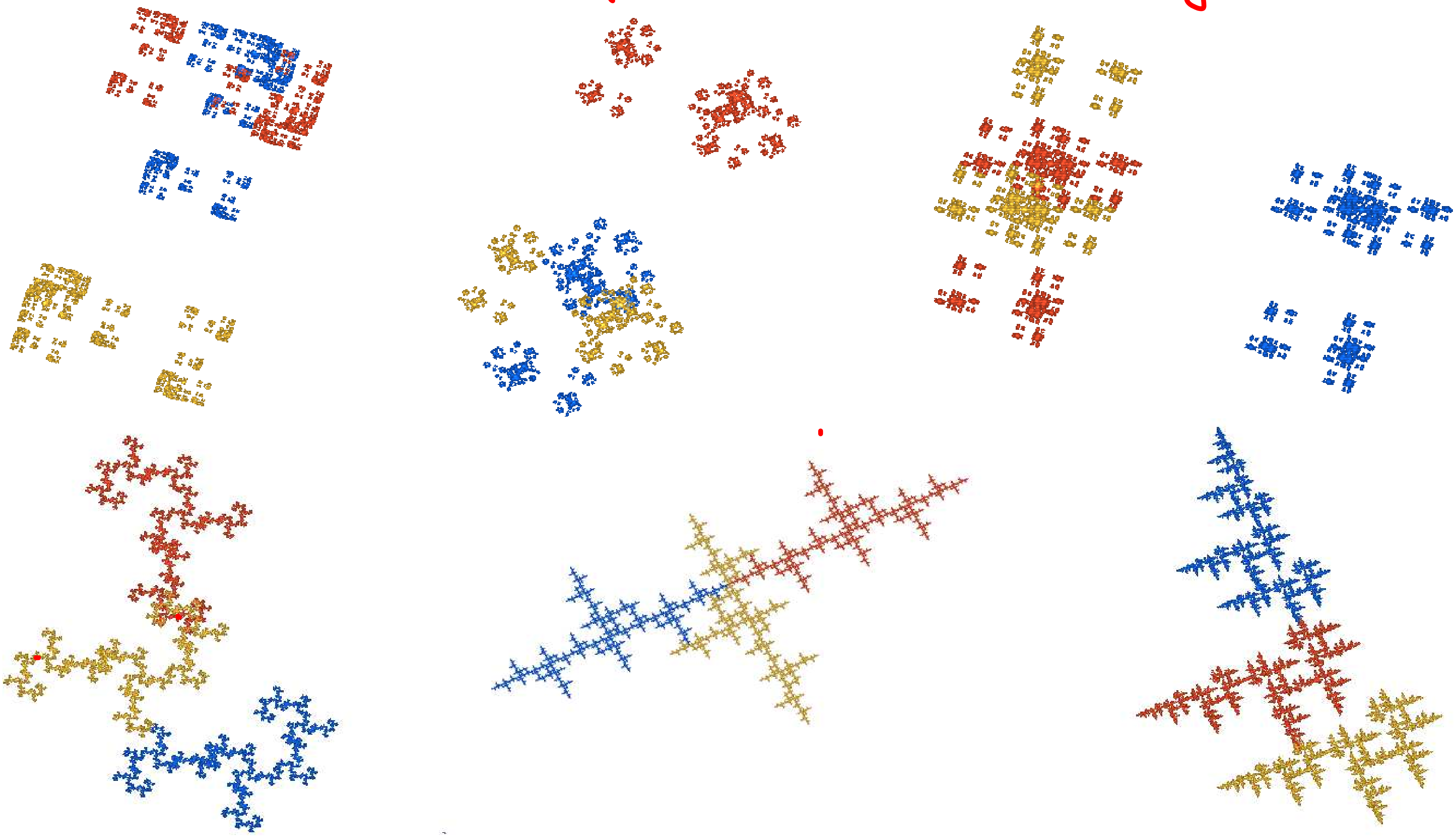




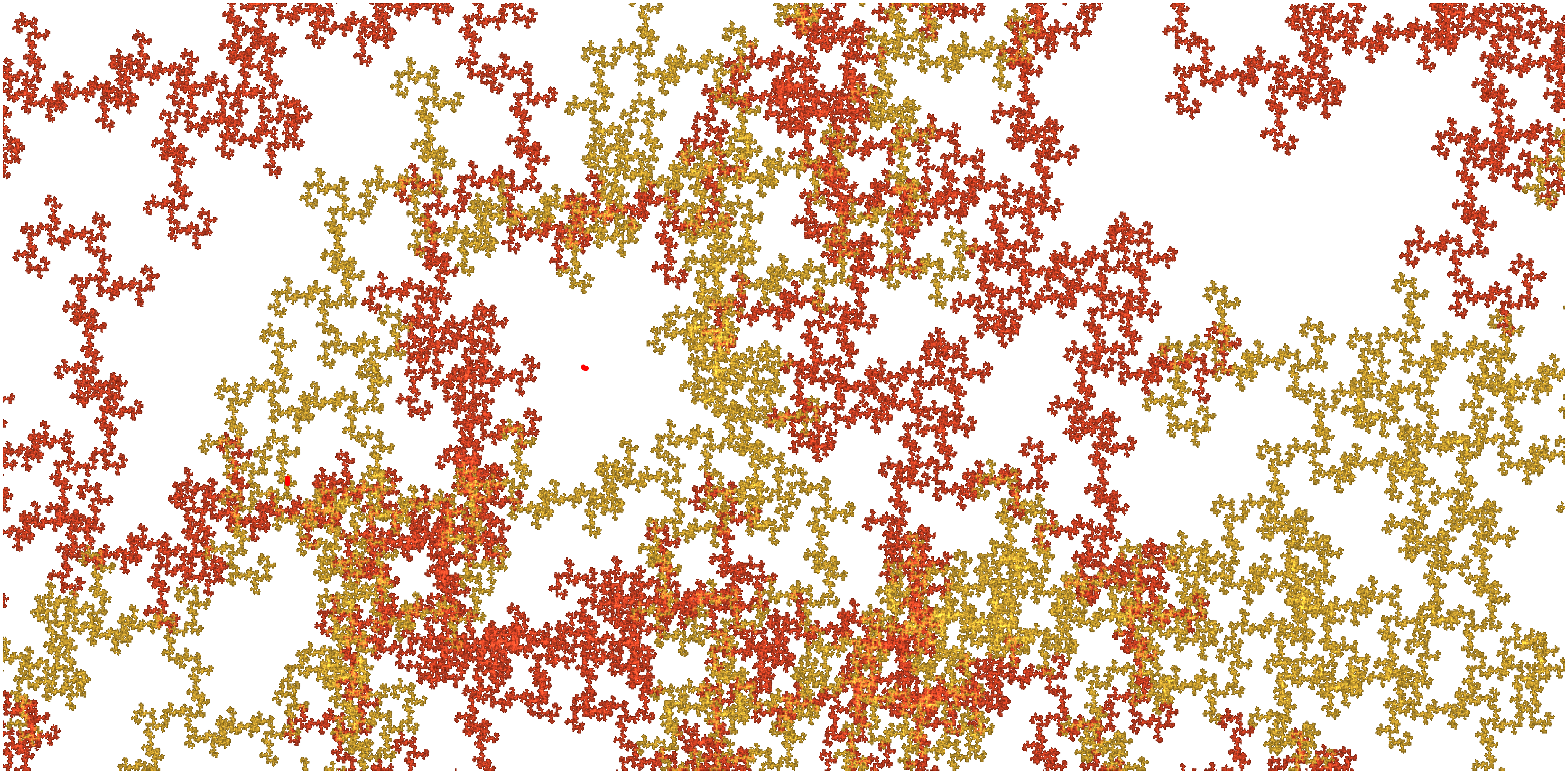
contains an irrational rotation between tiles,

(statistical circular symmetry, seen in spectrum)

4.2 New relatives of the Sierpinski triangle

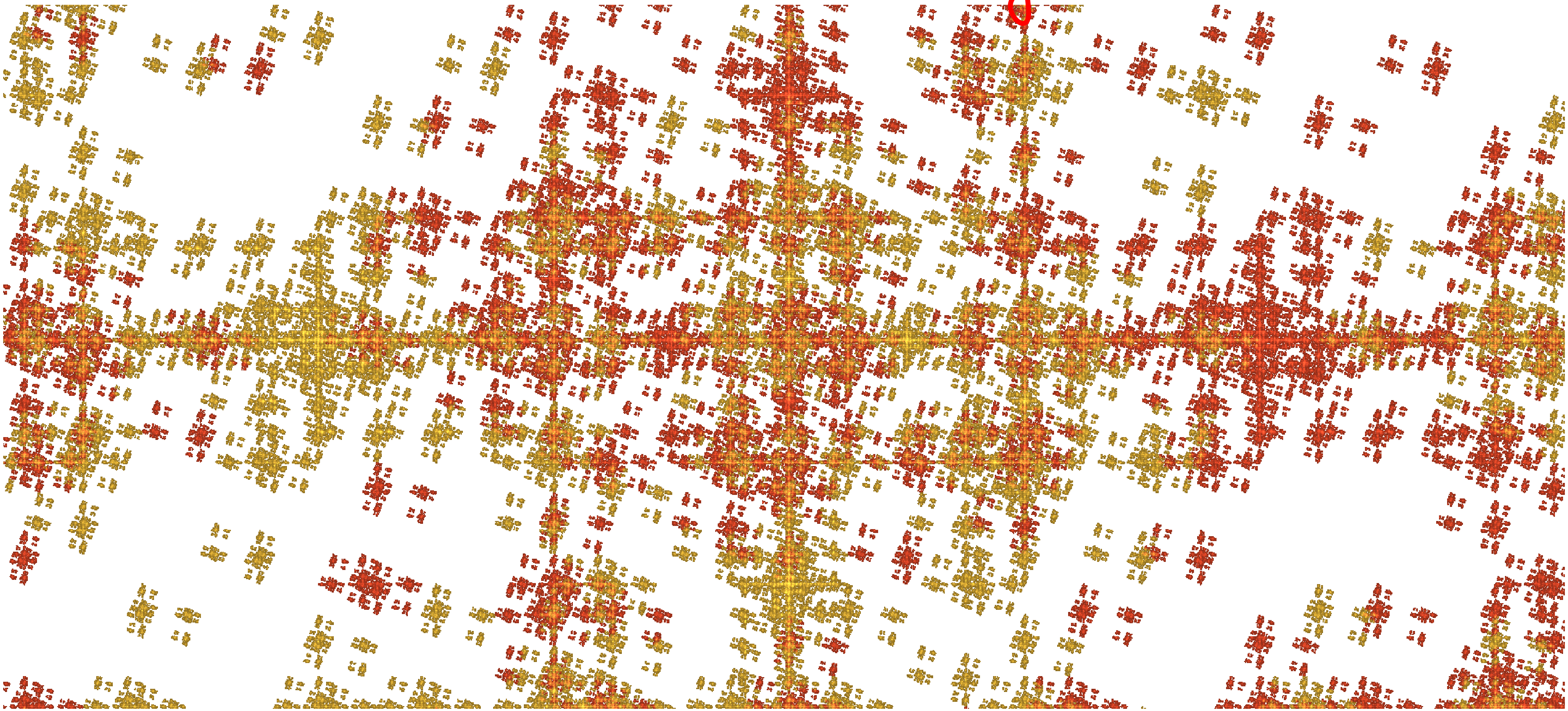


4.3 The local structure is only revealed under magnification



IFS: $f_1(z) = \frac{1}{2}(iz - 2 - i)$, $f_2(z) = \frac{1}{2}(iz - 3i)$, $f_3(z) = \frac{1}{2}(-z - 1 + i)$.

4.4 Even Cantor sets with many neighbors can have interesting metric structure



IFS: $f_1(z) = \frac{iz}{2}$, $f_2(z) = \frac{1}{2}(-iz+6)$, $f_2(z) = \frac{1}{2}(-iz+i)$.

Merci beaucoup !!