Integer multipliers of real polynomials without nonnegative roots

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Origin of the problem

Theorem (Laguerre (?), Poincaré (1883), Hurwitz (seminar talk 1904), Meissner (1911))

Let $f \in \mathbb{R}[X]$ be monic and suppose that $f(r) > 0$ for all $r \geq 0$. Then there exist $g, h \in \mathbb{R}_{>0}[X]$ such that

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Meissner: "positive representation of polynomials"

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Let $f \in \mathbb{R}[X]$ have a positive leading coefficient. Then

$$\delta(f) := \inf \{\deg(g) \ : \ g \in \mathbb{R}[X],\ gf \in \mathbb{R}_{>0}[X]\}$$

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The constants δ and δ_0 have been introduced by J.-P. Borel (1988). A polynomial g with $(*)$ is sometimes called a δ-multiplier (δ_0-multiplier, resp.) of f.

Theorem (Dancs (1964), Motzkin – Straus (1969))

For $b, c \in \mathbb{R}_{>0}$ *with* $b^2 < 4c$ *we have*

$$\delta(X^2 - bX + c) = \left\lfloor \frac{\pi}{\arcsin\sqrt{1 - \frac{b^2}{4c}}} \right\rfloor - 1$$

and

$$\delta_0(X^2 - bX + c) = \left\lceil \frac{\pi}{\arcsin\sqrt{1 - \frac{b^2}{4c}}} \right\rceil - 2\,.$$

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$$q = X^2 - (2r\cos\theta)X + r^2$$

with $r > 0$ *and* $\theta \in (0, \pi)$. *Further, let* $n \in \mathbb{N}$ *with*

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A multiplier of minimal degree may have negative coefficients: For

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we have

$$g \cdot \Phi_{30} = X^{15} + 1\,,$$

hence

$$m = \min\left\{\deg(h) \ : \ h \in \mathbb{R}[X] \setminus \{0\}\,,\ h \cdot \Phi_{30} \in \mathbb{R}_{\geq 0}[X]\right\} \leq \deg(g) = 7\,.$$

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Let ζ be a primitive 30-th root of unity, thus

$$q = (X - \zeta)(X - \overline{\zeta}) = X^2 - (2\cos(\pi/15))X + 1 \mid \Phi_{30} \mid X^{15} + 1$$

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$$q = (X - \zeta)(X - \overline{\zeta}) = X^2 - (2\cos(\pi/15))X + 1 \mid \Phi_{30} \mid X^{15} + 1$$

and the Theorem above yields $m = 7$. We conclude that g is a multiplier of minimal degree of Φ_{30}.

Theorem (Handelman (1985))

Let $f \in \mathbb{R}[X]$ have a positive leading coefficient and suppose that f does not have nonnegative roots. For every $p \in \mathbb{R}_{>0}[X]$ there exists some $n \in \mathbb{N}$ such that $p^n f$ has only positive coefficients.

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$$p = \sum_{i=0}^{m} a_i X^i = X^{m-1}\left(a_m X + \frac{a_{m-1}}{2}\right) + \sum_{i=1}^{m-2} X^i \left(\frac{a_{i+1}}{2} X + \frac{a_i}{2}\right) + \frac{a_1}{2} X + a_0$$

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For the quadratic polynomials with $b < 0$ compute

$$\nu_{X^2+bX+c}(r) := \min\left\{ n \in \mathbb{N} \;:\; (X+r)^n \cdot (X^2 + bX + c) \in \mathbb{R}_{>0}[X] \right\}.$$

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Lemma

Let $b, c, r \in \mathbb{R}_{>0}$ *with* $b^2 < 4c$. *For* $q := X^2 - bX + c$ *we can effectively compute* $\nu_q(r)$ *using the inequalities*

$$\max\left\{\frac{b}{r},\ \frac{br}{c},\ \delta(q) - 1\right\} < \nu_q(r)$$

and

$$\nu_q(r) \leq \min\left\{m \in \mathbb{N}\ :\ m > \max\left\{\frac{b}{r},\ \frac{br}{c},\ \frac{\beta + \sqrt{\beta^2 + A\gamma}}{Ar}\right\}\right\},$$

where we set $A := 4c - b^2$ *and*

$$\beta := br^2 + 2(b^2 - 2c)r + bc, \quad \gamma := r^4 + 4br^3 + 2(2b^2 - c)r^2 + 4bcr + c^2\,.$$

Further, we can effectively compute

$$\nu_{q,0}(r) := \min\{n \in \mathbb{N} \ : \ (X+r)^n q \in \mathbb{R}_{\geq 0}[X]\}$$

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Idea of the proof (based on work by Akiyama (2006)):

$$(X+r)^n \cdot q = r^n c + r^{n-1}(cn - br)X + \sum_{k=2}^{n} p_k X^k + (nr - b)X^{n+1} + X^{n+2}$$

with

$$p_k := \binom{n}{k-2} r^{n-k+2} - b \binom{n}{k-1} r^{n-k+1} + c \binom{n}{k} r^{n-k}$$

or

$$p_k = \binom{n}{k-1} r^{n-k} f(k) \qquad\qquad (2 \le k \le n)$$

with

$$f(k) := \frac{n+1-k}{k} c - br + \frac{k-1}{n+2-k} r^2$$

Observe

$$k(n-k+2)\, f(k) = g(k)$$

with

$$g(x) := \delta x^2 - ((2c+br)n + \sigma)x + c(n^2+3n+2)$$

and

$$\delta := r^2 + br + c \qquad \text{and} \qquad \sigma := r^2 + 2br + 3c\,.$$

For sufficiently large n the discriminant of g is negative, thus

$$f(k) > 0 \qquad (k \in \{2, \ldots, n\}).$$

Corollary

Let $r \in \mathbb{N}$, $\omega \in \mathbb{C} \setminus \mathbb{R}$ *and set*

$$q := (X - \omega)(X - \bar{\omega}).$$

Further, set $d_0(\omega) := d(\omega) := 0$ *if* $\Re(\omega) \leq 0$, *otherwise* $d_0(\omega) := \delta_0(q)$ *and* $d(\omega) := \delta(q)$.

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(i) *We can effectively compute a constant* $K_0(r, \omega) \in \mathbb{N}_0$ *and a monic integer polynomial* t *with* $d_0(\omega) \leq \deg(t) \leq K_0(r, \omega),\ t \cdot q \in \mathbb{R}_{\geq 0}[X],\ t(0) = r^{\deg(t)}$.

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Proof.

For $\Re(\omega) \leq 0$ we set $K(r, \omega) = K_0(r, \omega) := 0$ and $s = t = 1$, and for $\Re(\omega) > 0$ we apply the Lemma with $K(r, \omega) := \nu_q(r)$ and $K_0(r, \omega) := \nu_{q,0}(r)$. □

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Proof by a straightforward induction on the cardinality of $\mathrm{Z}_{f,+}$.

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Our result above immediately yields

Corollary

Let f be the minimal polynomial of the algebraic number $\alpha \neq 0$. Then α is positively algebraic if and only if there exists a monic integer polynomial t with $t(0) = 1$ and

$$\deg(t) \leq \sum_{\omega \in \mathrm{Z}_{f,+}} K_0(1, \omega)$$

such that the product tf has only nonnegative coefficients.

Example

For $q = X^2 - (8/5)X + 1$ we calculate $\delta(q) = \delta_0(q) = 3$. For a few natural integers r we compute the lower and upper bounds $L(r)$ and $U(r)$ for $\nu_q(r)$ and then the constants $\nu_q(r)$ and $\nu_{q,0}(r)$:

Example

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r	$L(r)$	$U(r)$	$\nu_q(r)$	$\nu_{q,0}(r)$
1	4	9	9	7
2	4	10	9	9
3	5	12	12	11
4	7	14	14	14
5	9	17	17	16

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we find:

$$p_1 = X^9 + \frac{27}{5}X^8 + \frac{54}{5}X^7 + \frac{42}{5}X^6 + \frac{42}{5}X^3 + \frac{54}{5}X^2 + \frac{27}{5}X + 1\,,$$

$$p_2 = X^{11} + \frac{82}{5}X^{10} + \frac{581}{5}X^9 + \frac{2298}{5}X^8 + \frac{5424}{5}X^7 + \frac{7392}{5}X^6 + \frac{4704}{5}X^5 + \frac{192}{5}X^4 + \frac{1536}{5}X^3 + \frac{7168}{5}X^2 + \frac{7424}{5}X + 512$$

$$p_3 = X^{13} + \frac{157}{5}X^{12} + \frac{2216}{5}X^{11} + 3696X^{10} + 20097X^9 + 73953X^8 + \frac{919512}{5}X^7 + \frac{1475496}{5}X^6 + 264627X^5 + 72171X^4 + \frac{1102248}{5}X^2 + \frac{1830519}{5}X + 177147$$

$$
\begin{aligned}
p_4 = X^{16} &+ \frac{272}{5}X^{15} + \frac{6837}{5}X^{14} + \frac{105112}{5}X^{13} + \frac{1102192}{5}X^{12} \\
&+ \frac{8316672}{5}X^{11} + \frac{46382336}{5}X^{10} + \frac{192997376}{5}X^{9} \\
&+ \frac{595685376}{5}X^{8} + \frac{1330774016}{5}X^{7} + \frac{2033647616}{5}X^{6} \\
&+ \frac{1860698112}{5}X^{5} \\
&+ \frac{667942912}{5}X^{4} + \frac{117440512}{5}X^{3} \\
&+ \frac{1459617792}{5}X^{2} + \frac{2550136832}{5}X + 268435456
\end{aligned}
$$

$$p_5 = X^{18} + \frac{392}{5}X^{17} + 2873X^{16} + 65280X^{15} + 1028500X^{14}$$

$$+11900000X^{13} + 104422500X^{12} + 707200000X^{11}$$

$$+3722468750X^{10} + 15193750000X^{9} + 47480468750X^{8}$$

$$+110500000000X^{7} + 181289062500X^{6}$$

$$+185937500000X^{5} + 83007812500X^{4}$$

$$+103759765625X^{2} + 244140625000X + 152587890625$$

Concluding remarks

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- Examples suggest that the ν-values always lie close to the upper bounds given above. It might be an interesting problem to quantify this observation.

- Let p be a CNS polynomial. Akiyama (2001) predicted that the leading coefficient of the canonical representative of -1 w.r.t. p equals 1. Our result shows that there is a monic polynomial $t \in \mathbb{Z}[X]$ with the properties

$$t \cdot p \in \mathbb{N}_0[X] \qquad \text{and} \qquad t(0) = 1\,.$$

To prove Akiyama's Conjecture it suffices to show that apart form the leading and the constant terms all coefficients of tp belong $\{0, \ldots, p(0) - 1\}$. Unfortunately, our approach here does not provide further progress in this direction.

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  - Factors with negative coefficients of polynomials with positive coefficients for instance,
    $$X^3 + X^2 + 2X + 8 = (X^2 - X + 4) \cdot (X + 2)$$
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- Our results here complete the proof of Theorem 21 in

B., On expanding real polynomials with a given factor (2013):

In its proof it was not shown that an integer multiplier can be found with leading coefficient 1.