On the magic of some families of fractal dendrites

Ligia Loretta Cristea

FWF stand-alone project P27050-N26

Universität Graz

joint work with Bertran Steinsky and Gunther Leobacher

Numeration 2018, Paris, May 22-25, 2018

- LLC, B. Steinsky, **Paths of infinite length in** 4×4 **- labyrinth fractals**, *Geometriae Dedicata (2009)*
- LLC, B. Steinsky, **Curves of Infinite Length in Labyrinth Fractals**, *Proceedings of the Edinburgh Math. Soc. (2010)*
- LLC, B. Steinsky, **Mixed labyrinth fractals**, *J. Topology and its Applications* (2017)
- LLC, G. Leobacher, **A note on lengths of arcs in mixed labyrinth fractals**, *Monatshefte f. Mathematik* (2017)
- LLC, G. Leobacher, **Supermixed labyrinth fractals**, submitted 2018

What is a labyrinth fractal?

- 4 "doors": golden, very "small"
- from any of the "doors" a path is starting
- all "doors" are connected
- infinitely many, infinitely small rooms:
- only one path between any two (rooms)
- this unique path is infinitely long (under certain conditions)

What is a labyrinth fractal?

- 4 "doors": golden, very "small"
- from any of the "doors" a path is starting
- all "doors" are connected
- infinitely many, infinitely small rooms:
- only one path between any two (rooms)
- this unique path is infinitely long (under certain conditions)

What is a labyrinth fractal?

- 4 "doors": golden, very "small"
- from any of the "doors" a path is starting
- all "doors" are connected
- infinitely many, infinitely small rooms:
- only one path between any two (rooms)
- this unique path is infinitely long (under certain conditions)

What is a labyrinth fractal?

- 4 "doors": golden, very "small"
- from any of the "doors" a path is starting
- all "doors" are connected
- infinitely many, infinitely small rooms:
- only one path between any two (rooms)
- this unique path is infinitely long (under certain conditions)

What is a labyrinth fractal?

- 4 "doors": golden, very "small"
- from any of the "doors" a path is starting
- all "doors" are connected
- infinitely many, infinitely small rooms:
- only one path between any two (rooms)
- this unique path is infinitely long (under certain conditions)

What is a labyrinth fractal?

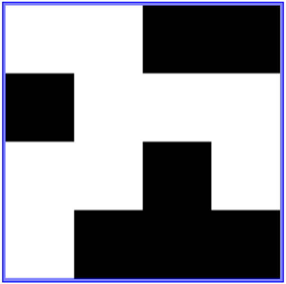
- 4 "doors": golden, very "small"
- from any of the "doors" a path is starting
- all "doors" are connected
- infinitely many, infinitely small rooms:
- only one path between any two (rooms)
- this unique path is infinitely long (under certain conditions)

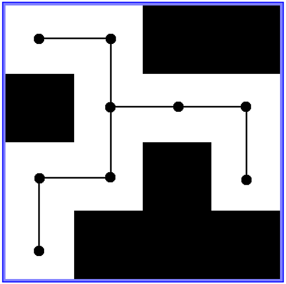
What is a labyrinth fractal?

- 4 "doors": golden, very "small"
- from any of the "doors" a path is starting
- all "doors" are connected
- infinitely many, infinitely small rooms:
- only one path between any two (rooms)
- this unique path is infinitely long (under certain conditions)

(Labyrinth) patterns. The graph of a (labyrinth) pattern

A 4×4 (labyrinth) pattern and its graph

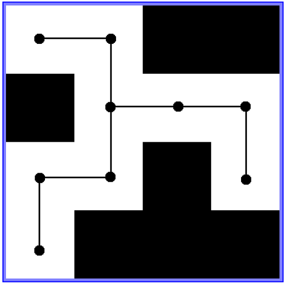




What is a labyrinth pattern?

Property 1 (The Tree Property)

The graph of the labyrinth pattern



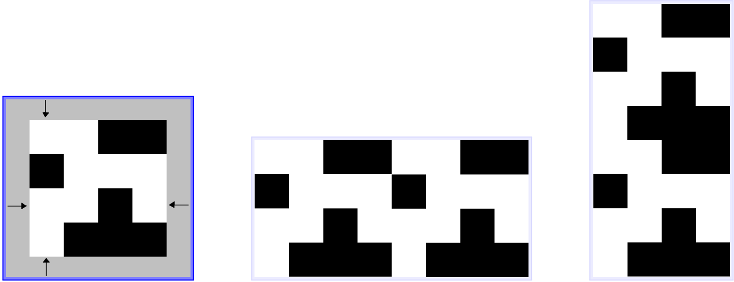
Property

The graph of the labyrinth pattern is a tree. (the Tree Property)

Property 2 (The Exits Property)

Property

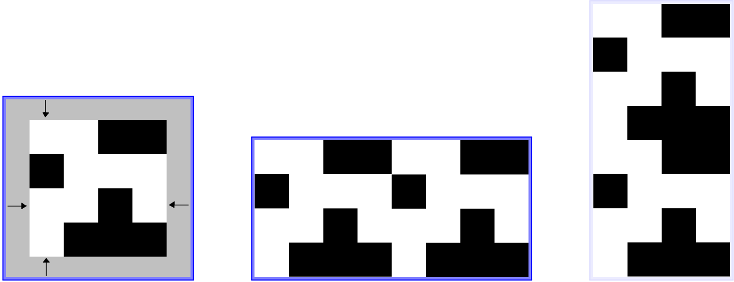
There is exactly one horizontal and exactly one vertical exit pair in the labyrinth pattern. (the Exits Property)



Property 2 (The Exits Property)

Property

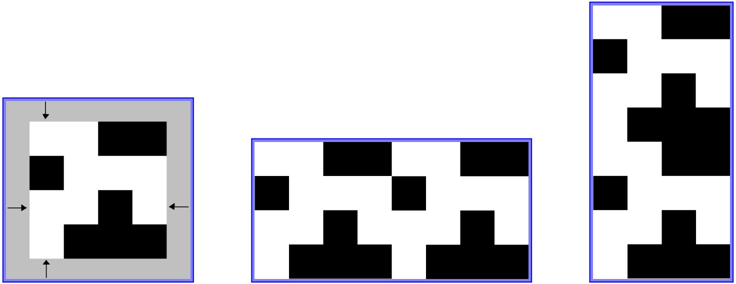
There is exactly one horizontal and exactly one vertical exit pair in the labyrinth pattern. (the Exits Property)



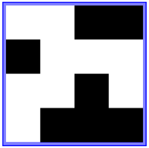
Property 2 (The Exits Property)

Property

There is exactly one horizontal and exactly one vertical exit pair in the labyrinth pattern. (the Exits Property)

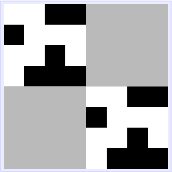


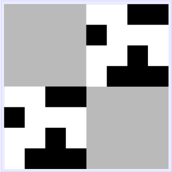
Property 3 (The Corner Property)



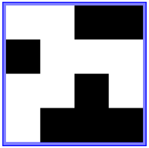
Property

If there is a white square at a corner of the labyrinth pattern, then there is no white square at the diagonally opposite corner of the labyrinth pattern. (the Corner Property)



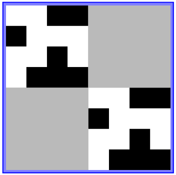


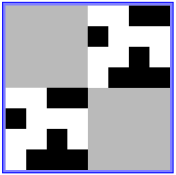
Property 3 (The Corner Property)



Property

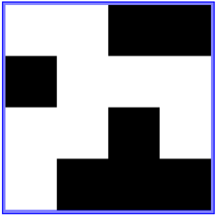
If there is a white square at a corner of the labyrinth pattern, then there is no white square at the diagonally opposite corner of the labyrinth pattern. (the Corner Property)

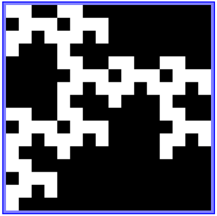


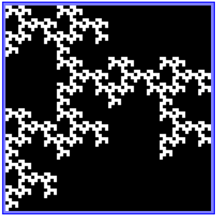


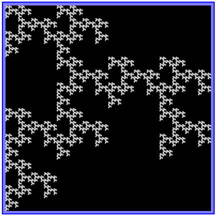
Construction of a labyrinth fractal

A 4×4-labyrinth pattern/set.





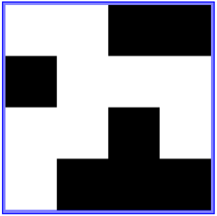


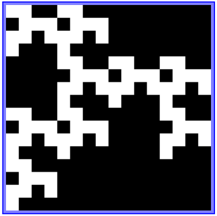


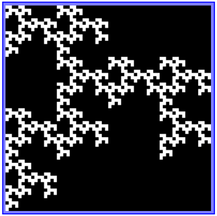
. . . labyrinth fractal

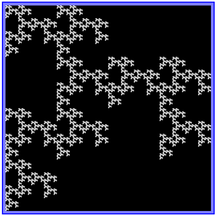
Construction of a labyrinth fractal

A 4×4-labyrinth pattern/set.







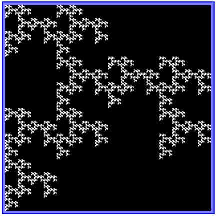


... labyrinth fractal

Dendrites

Theorem

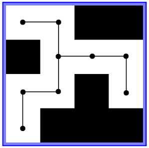
For **all** *$m \times m$ labyrinth patterns, the constructed self-similar fractal L is a dendrite.*



Dendrite

A *dendrite* is a connected and locally connected compact Hausdorff space that contains no simple closed curve.

A Fourth Property



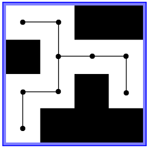
Horizontally Blocked

A labyrinth pattern is called *horizontally blocked* if the row (of squares) from the left exit to the right exit contains at least one black square.

Vertically Blocked

A labyrinth pattern is called *vertically blocked* if the column (of squares) from the top exit to the bottom exit contains at least one black square.

A Fourth Property



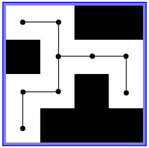
Horizontally Blocked

A labyrinth pattern is called *horizontally blocked* if the row (of squares) from the left exit to the right exit contains at least one black square.

Vertically Blocked

A labyrinth pattern is called *vertically blocked* if the column (of squares) from the top exit to the bottom exit contains at least one black square.

A Fourth Property



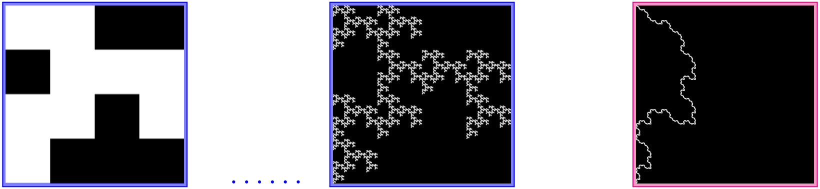
Horizontally Blocked

A labyrinth pattern is called *horizontally blocked* if the row (of squares) from the left exit to the right exit contains at least one black square.

Vertically Blocked

A labyrinth pattern is called *vertically blocked* if the column (of squares) from the top exit to the bottom exit contains at least one black square.

Self-similar labyrinth fractals. Main Result



Theorem

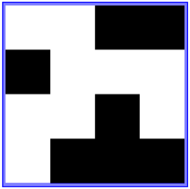
Let L_∞ be the (self-similar) labyrinth fractal generated by a horizontally and vertically blocked $m \times m$-labyrinth pattern.

(a) *Between any two points in L_∞ there is a unique arc a.*

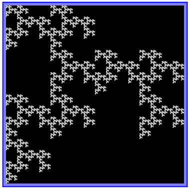
(b) *The length of a is infinite and $\dim_B(a) = \frac{\log r}{\log m}$.*

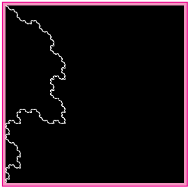
(c) *The set of all points, at which no tangent to a exists, is dense in a.*

Self-similar labyrinth fractals. Main Result



. . . .





Theorem

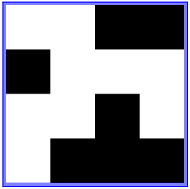
Let L_∞ be the (self-similar) labyrinth fractal generated by a horizontally and vertically blocked $m \times m$-labyrinth pattern.

(a) *Between any two points in L_∞ there is a unique arc a.*

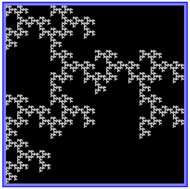
(b) *The length of a is infinite and $\dim_B(a) = \frac{\log r}{\log m}$.*

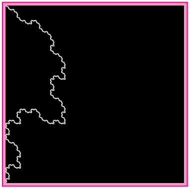
(c) *The set of all points, at which no tangent to a exists, is dense in a.*

Self-similar labyrinth fractals. Main Result



....





Theorem

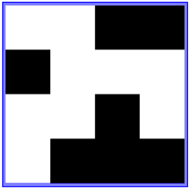
Let L_∞ be the (self-similar) labyrinth fractal generated by a horizontally and vertically blocked $m \times m$-labyrinth pattern.

(a) *Between any two points in L_∞ there is a unique arc a.*

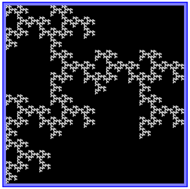
(b) *The length of a is infinite and* $\dim_B(a) = \frac{\log r}{\log m}$.

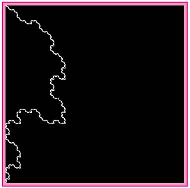
(c) *The set of all points, at which no tangent to a exists, is dense in a.*

Self-similar labyrinth fractals. Main Result



. . . .





Theorem

Let L_∞ be the (self-similar) labyrinth fractal generated by a horizontally and vertically blocked $m \times m$-labyrinth pattern.

(a) *Between any two points in L_∞ there is a unique arc a.*

(b) *The length of a is infinite and $\dim_B(a) = \frac{\log r}{\log m}$.*

(c) *The set of all points, at which no tangent to a exists, is dense in a.*

Famous Theorems used for the proofs

- **Jordan Curve** Theorem
- **Hahn-Mazurkiewicz-Sierpiński** Theorem
- **Perron-Frobenius** Theorem
- a *labyrinth version* of the **Steinhaus Chessboard** Theorem

Mixed labyrinth fractals

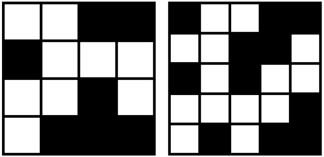


Figure: Two labyrinth patterns, $\mathcal{A}_1$ (a 4-pattern) and $\mathcal{A}_2$ (a 5-pattern)

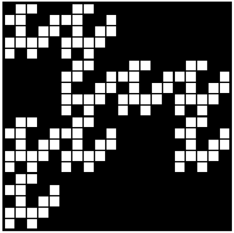


Figure: The mixed (labyrinth) set $\mathcal{W}_2$, constructed based on the above patterns $\mathcal{A}_1$ and $\mathcal{A}_2$, that can also be viewed as a 20-pattern

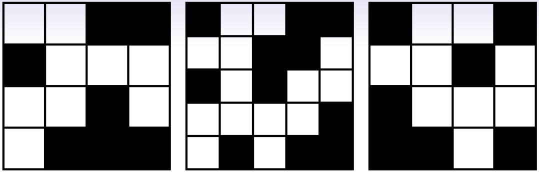


Figure: Labyrinth patterns: $\mathcal{A}_1$, $\mathcal{A}_2$ (as before), and $\mathcal{A}_3$ (4×4)

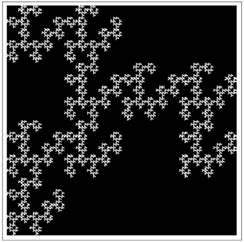


Figure: The mixed (labyrinth) set of level 4 defined by a sequence $\{\mathcal{A}_k\}_{k\geq 1}$ where the first three patterns are $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, respectively, shown above, and the fourth is $\mathcal{A}_1$

Topological properties of mixed labyrinth fractals

Lemma

Let $\{\mathcal{A}_k\}_{k=1}^{\infty}$ be a sequence of non-empty patterns, $m_k \geq 3$, and $n \geq 1$. If $\mathcal{A}_1, \dots \mathcal{A}_n$ are labyrinth patterns, then $\mathcal{W}_n$ is an $m(n) \times m(n)$-labyrinth set (i.e., it has the Tree Property, Exits Property, Corner property), for all $n \geq 1$, where $m(n) = \prod_{k=1}^{n} m_k$.

We call the limit set $L_\infty = \bigcap_{n\geq 1} \bigcup_{W \in \mathcal{W}_n} W$ the mixed labyrinth fractal generated by $\{\mathcal{A}_k\}_{k=1}^{\infty}$.

Theorem

Let $\{\mathcal{A}_k\}_{k=1}^{\infty}$ be a sequence of labyrinth patterns, $m_k \geq 3$, for all $k \geq 1$. Then L_∞ is a dendrite.

Topological properties of mixed labyrinth fractals

Lemma

Let $\{\mathcal{A}_k\}_{k=1}^{\infty}$ be a sequence of non-empty patterns, $m_k \geq 3$, and $n \geq 1$. If $\mathcal{A}_1, \ldots \mathcal{A}_n$ are labyrinth patterns, then $\mathcal{W}_n$ is an $m(n) \times m(n)$-labyrinth set (i.e., it has the Tree Property, Exits Property, Corner property), for all $n \geq 1$, where $m(n) = \prod_{k=1}^{n} m_k$.

We call the limit set $L_\infty = \bigcap_{n\geq 1} \bigcup_{W \in \mathcal{W}_n} W$ the mixed labyrinth fractal generated by $\{\mathcal{A}_k\}_{k=1}^{\infty}$.

Theorem

Let $\{\mathcal{A}_k\}_{k=1}^{\infty}$ be a sequence of labyrinth patterns, $m_k \geq 3$, for all $k \geq 1$. Then L_∞ is a dendrite.

Topological properties of mixed labyrinth fractals

Lemma

Let $\{\mathcal{A}_k\}_{k=1}^{\infty}$ be a sequence of non-empty patterns, $m_k \geq 3$, and $n \geq 1$. If $\mathcal{A}_1, \dots \mathcal{A}_n$ are labyrinth patterns, then $\mathcal{W}_n$ is an $m(n) \times m(n)$-labyrinth set (i.e., it has the Tree Property, Exits Property, Corner property), for all $n \geq 1$, where $m(n) = \prod_{k=1}^{n} m_k$.

We call the limit set $L_\infty = \bigcap_{n \geq 1} \bigcup_{W \in \mathcal{W}_n} W$ the mixed labyrinth fractal generated by $\{\mathcal{A}_k\}_{k=1}^{\infty}$.

Theorem

Let $\{\mathcal{A}_k\}_{k=1}^{\infty}$ be a sequence of labyrinth patterns, $m_k \geq 3$, for all $k \geq 1$. Then L_∞ is a dendrite.

The construction of the path between exits

Example: The path between the bottom exit and the right exit

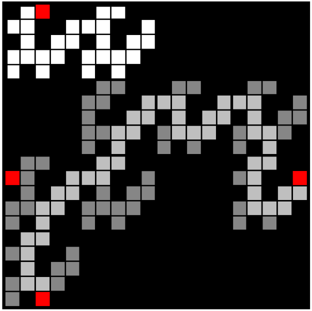


Figure: The set $\mathcal{W}_2$ constructed with the patterns $\mathcal{A}_1$ and $\mathcal{A}_2$ shown before, and the path from the bottom exit to the right exit of $\mathcal{W}_2$ (in lighter gray).

One can check that ◲$(2) = 48$.

The idea of the construction described in the following works for all mixed labyrinth fractals.

Paths in mixed labyrinth sets. Paths in patterns

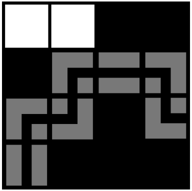


Figure: The path from the bottom exit to the right exit of $\mathcal{A}_1$

- first, we find the path between the bottom and the right exit of $\mathcal{W}_1$
- then we denote each white square in the path according to its neighbours within the path: there are 6 possible types of squares: , , , , , and -*square*

Paths in mixed labyrinth sets. Paths in patterns

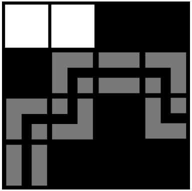


Figure: The path from the bottom exit to the right exit of $\mathcal{A}_1$

- first, we find the path between the bottom and the right exit of $\mathcal{W}_1$
- then we denote each white square in the path according to its neighbours within the path: there are 6 possible types of squares: ◫, ⊟, ◳, ◰, ◱, and ◲-*square*

Paths in mixed labyrinth sets. Paths in patterns

In order to obtain the ◲-path in $\mathcal{G}(\mathcal{W}_2)$, we replace each ◧-square of the path in $\mathcal{G}(\mathcal{W}_1)$ with the ◧-path in $\mathcal{G}(\mathcal{A}_2)$.
Analogously, we do this for the other marked white squares.

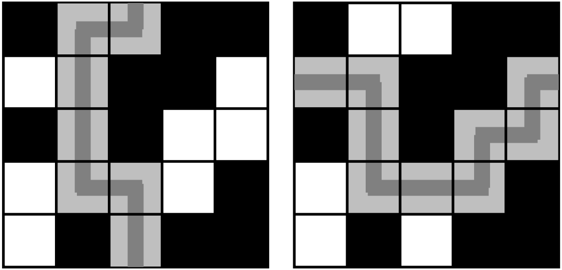
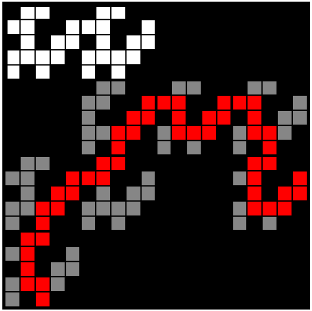


Figure: Paths from bottom to top and from left to right exit of $\mathcal{A}_2$

Paths in mixed labyrinth sets



In general, for any pair of exits and $n \geq 1$, we replace each marked white square in the path of $\mathcal{G}(\mathcal{W}_n)$ by its corresponding path in $\mathcal{G}(\mathcal{A}_{n+1})$ and obtain the path of $\mathcal{G}(\mathcal{W}_{n+1})$.

Paths in mixed labyrinth sets

Let $\{\mathcal{A}_k\}_{k\geq 1}$ be a sequence of labyrinth patterns, that defines the sequence $\{\mathcal{W}_n\}_{n\geq 1}$ of mixed labyrinth sets.

Proposition

There exist non-negative 6×6*-matrices* M_k, $k = 1, 2, \ldots,$ *such that for all* $n \geq 1$, *and for* $M(n) = M_1 \cdot M_2 \cdot \ \ldots \ \cdot M_n$, *the element in row x and column y of* $M(n)$ *is the number of y-squares in the x-path in* $\mathcal{G}(\mathcal{W}_n)$, *for* $x, y \in \{\text{┃}, \text{━}, \text{┗}, \text{┏}, \text{┓}, \text{┛}\}$.

Furthermore,

$$\begin{pmatrix} \text{┃}(n) \\ \text{━}(n) \\ \text{┗}(n) \\ \text{┏}(n) \\ \text{┓}(n) \\ \text{┛}(n) \end{pmatrix} = M(n) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Paths in mixed labyrinth sets

Let $\{\mathcal{A}_k\}_{k\geq 1}$ be a sequence of labyrinth patterns, that defines the sequence $\{\mathcal{W}_n\}_{n\geq 1}$ of mixed labyrinth sets.

Proposition

There exist non-negative 6×6*-matrices* M_k, $k = 1, 2, \dots,$ *such that for all* $n \geq 1$, *and for* $M(n) = M_1 \cdot M_2 \cdot \ \dots \ \cdot M_n$, *the element in row x and column y of* $M(n)$ *is the number of y-squares in the x-path in* $\mathcal{G}(\mathcal{W}_n)$, *for* $x, y \in \{\text{◫}, \text{⊟}, \text{◰}, \text{◲}, \text{◱}, \text{◳}\}$.
Furthermore,

$$\begin{pmatrix} \text{◫}(n) \\ \text{⊟}(n) \\ \text{◰}(n) \\ \text{◲}(n) \\ \text{◱}(n) \\ \text{◳}(n) \end{pmatrix} = M(n) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Paths in mixed labyrinth sets

Sketch of the proof

For $k \geq 1$, we define the matrix M_k (the path matrix of $\mathcal{A}_k$):

- the columns of M_k from left to right and the rows of M_k from top to bottom correspond to , , , , , and , (ordered set of indices)
- the element in row x and column y of M_k is the number of y-squares in the x-path in $\mathcal{G}(\mathcal{A}_k)$.

One can easily check that the **matrix multiplication reflects the substitution of paths**.

(Proof by induction)

Remark. The proposition yields

- in the *self-similar* case $M(n) = M^n$
- in the *mixed* case $M(n+1) = M(n) \cdot M_{n+1}$

Paths in mixed labyrinth sets

Sketch of the proof

For $k \geq 1$, we define the matrix M_k (the path matrix of $\mathcal{A}_k$):

- the columns of M_k from left to right and the rows of M_k from top to bottom correspond to ◫, ⊟, └, ┌, ┐, and ┘, (ordered set of indices)
- the element in row x and column y of M_k is the number of y-squares in the x-path in $\mathcal{G}(\mathcal{A}_k)$.

One can easily check that the **matrix multiplication reflects the substitution of paths**.

(Proof by induction)

Remark. The proposition yields

- in the *self-similar* case $M(n) = M^n$
- in the *mixed* case $M(n+1) = M(n) \cdot M_{n+1}$

Paths in mixed labyrinth sets

Sketch of the proof

For $k \geq 1$, we define the matrix M_k (the path matrix of $\mathcal{A}_k$):

- the columns of M_k from left to right and the rows of M_k from top to bottom correspond to ◫, ⊟, ◰, ◲, ◳, and ◱, (ordered set of indices)
- the element in row x and column y of M_k is the number of y-squares in the x-path in $\mathcal{G}(\mathcal{A}_k)$.

One can easily check that the **matrix multiplication reflects the substitution of paths**.

(Proof by induction)

Remark. The proposition yields

- in the *self-similar* case $M(n) = M^n$
- in the *mixed* case $M(n+1) = M(n) \cdot M_{n+1}$

Paths in mixed labyrinth sets

Sketch of the proof

For $k \geq 1$, we define the matrix M_k (the path matrix of $\mathcal{A}_k$):

- the columns of M_k from left to right and the rows of M_k from top to bottom correspond to , , , , , and , (ordered set of indices)
- the element in row x and column y of M_k is the number of y-squares in the x-path in $\mathcal{G}(\mathcal{A}_k)$.

One can easily check that the **matrix multiplication reflects the substitution of paths**.

(Proof by induction)

Remark. The proposition yields

- in the *self-similar* case $M(n) = M^n$
- in the *mixed* case $M(n+1) = M(n) \cdot M_{n+1}$

Paths in mixed labyrinth sets

Sketch of the proof

For $k \geq 1$, we define the matrix M_k (the path matrix of $\mathcal{A}_k$):

- the columns of M_k from left to right and the rows of M_k from top to bottom correspond to ◫, ⊟, └, ┌, ┐, and ┘, (ordered set of indices)
- the element in row x and column y of M_k is the number of y-squares in the x-path in $\mathcal{G}(\mathcal{A}_k)$.

One can easily check that the **matrix multiplication reflects the substitution of paths**.

(Proof by induction)

Remark. The proposition yields

- in the *self-similar* case $M(n) = M^n$
- in the *mixed* case $M(n+1) = M(n) \cdot M_{n+1}$

Mixed labyrinth fractals generated by special cross patterns

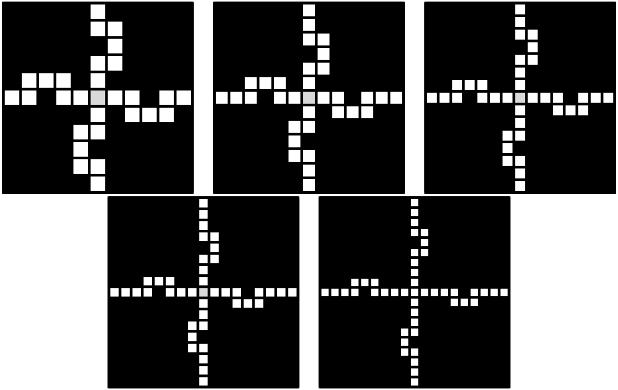


Figure: Example: the first five elements of a sequence of special cross patterns, where $m_k = 2k + 9$, and $a_k = k + 4$

Theorem

There **exist** *sequences* $\{\mathcal{A}_k\}_{k=1}^{\infty}$ *of both horizontally and vertically blocked labyrinth patterns, such that the limit set* L_∞ *has the property that for any two points in* L_∞ *the length of the arc* $a \subset L_\infty$ *that connects them is* **finite**. *For almost all points* $x_0 \in a$ *(with respect to the length) there exists the tangent at* x_0 *to the arc* a.

Proposition

There exist sequences $\{\mathcal{A}_k\}_{k=1}^{\infty}$ *of (both horizontally and vertically) blocked labyrinth patterns, such that the limit set* L_∞ *has the property that for any two points in* L_∞ *the length of the arc* $a \subset L_\infty$ *that connects them is* **infinite**.

Theorem

There **exist** *sequences* $\{\mathcal{A}_k\}_{k=1}^{\infty}$ *of both horizontally and vertically blocked labyrinth patterns, such that the limit set* L_∞ *has the property that for any two points in* L_∞ *the length of the arc* $a \subset L_\infty$ *that connects them is* **finite**. *For almost all* points $x_0 \in a$ *(with respect to the length) there exists the tangent at* x_0 *to the arc* a.

Proposition

There exist sequences $\{\mathcal{A}_k\}_{k=1}^{\infty}$ *of (both horizontally and vertically) blocked labyrinth patterns, such that the limit set* L_∞ *has the property that for any two points in* L_∞ *the length of the arc* $a \subset L_\infty$ *that connects them is* **infinite**.

Theorem

*There **exist** sequences $\{\mathcal{A}_k\}_{k=1}^{\infty}$ of both horizontally and vertically blocked labyrinth patterns, such that the limit set L_∞ has the property that for any two points in L_∞ the length of the arc $a \subset L_\infty$ that connects them is **finite**. For almost all points $x_0 \in a$ (with respect to the length) there exists the tangent at x_0 to the arc a.*

Proposition

*There exist sequences $\{\mathcal{A}_k\}_{k=1}^{\infty}$ of (both horizontally and vertically) blocked labyrinth patterns, such that the limit set L_∞ has the property that for any two points in L_∞ the length of the arc $a \subset L_\infty$ that connects them is **infinite**.*

NEW! Very recent results: On arclength in **mixed** labyrinth fractals

Theorem

Let $\{\mathcal{A}_k\}_{k\geq 1}$ be a sequence of horizontally and vertically blocked labyrinth patterns, such that the corresponding sequence of widths $\{m_k\}_{k\geq 1}$ satisfies the condition

$$\sum_{k\geq 1} \frac{1}{m_k} = \infty.$$

Then, for all $x, y \in L_\infty$ with $x \neq y$ the arc in L_∞ that connects x and y has infinite length.

Remark. The above condition is sufficient, but not necessary in order to obtain a labyrinth fractal with all arcs having infinite length.

NEW! Very recent results: On arclength in **mixed** labyrinth fractals

Theorem

Let $\{\mathcal{A}_k\}_{k\geq 1}$ be a sequence of horizontally and vertically blocked labyrinth patterns, such that the corresponding sequence of widths $\{m_k\}_{k\geq 1}$ satisfies the condition

$$\sum_{k\geq 1} \frac{1}{m_k} = \infty.$$

Then, for all $x, y \in L_\infty$ with $x \neq y$ the arc in L_∞ that connects x and y has infinite length.

Remark. The above condition is sufficient, but not necessary in order to obtain a labyrinth fractal with all arcs having infinite length.

Very recent results: **Supermixed** labyrinth fractals

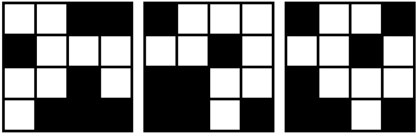


Figure: Three labyrinth patterns, from left to right: the unique pattern in $\{\mathcal{A}_{1,1}\} \in \widetilde{\mathcal{A}}_1$, followed by the (two) patterns $\mathcal{A}_{2,1}, \mathcal{A}_{2,2} \in \widetilde{\mathcal{A}}_2$, $m_1 = m_2 = 4$

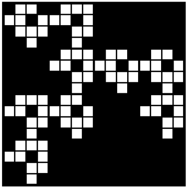


Figure: The (width-homogeneous) **supermixed** labyrinth set $\mathcal{W}_2$, constructed with the help of the above patterns from $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$, that can also be viewed as a 16-pattern

Very recent results: paths in **supermixed** labyrinth sets

Theorem

For all $n \geq 1$,

$$M(n) = Q_{n,1} + \cdots + Q_{n,s_{n+1}}, \quad s_{n+1} = \#(\widetilde{\mathcal{A}}_{n+1})$$

and

$$M(n+1) = \sum_{h=1}^{s_{n+1}} Q_{n,h} \cdot M_{n+1,h}, \quad Q_{n,h} = (q_{i,j}^{n,h})$$

where $q_{i,j}^{n,h}$ *is the number of j-squares in the path of type i in* $\mathcal{G}(\mathcal{W}_n)$ *which at the next step are "substituted" according to the pattern* $\mathcal{A}_{n+1,h}$.

Remark. For $s_{n+1} = 1$, (for some $n \geq 1$), we have $M(n) = Q_{n,1}$ and thus we *recover* the formula $M(n+1) = M(n) \cdot M_{n+1}$ proven earlier for mixed labyrinth fractals.

Very recent results: paths in **supermixed** labyrinth sets

Theorem

For all $n \geq 1$,

$$M(n) = Q_{n,1} + \cdots + Q_{n,s_{n+1}}, \quad s_{n+1} = \#(\widetilde{\mathcal{A}}_{n+1})$$

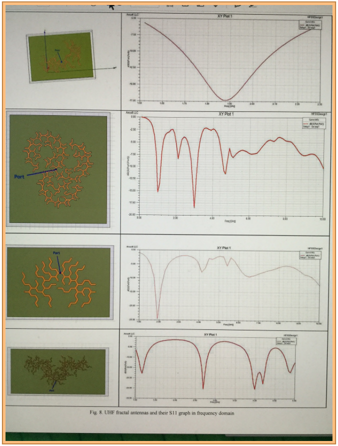
and

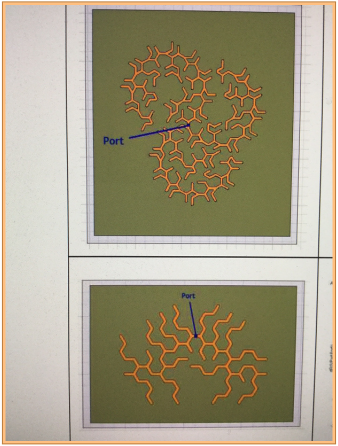
$$M(n+1) = \sum_{h=1}^{s_{n+1}} Q_{n,h} \cdot M_{n+1,h}, \quad Q_{n,h} = (q_{i,j}^{n,h})$$

where $q_{i,j}^{n,h}$ *is the number of j-squares in the path of type i in* $\mathcal{G}(\mathcal{W}_n)$ *which at the next step are "substituted" according to the pattern* $\mathcal{A}_{n+1,h}$.

Remark. For $s_{n+1} = 1$, (for some $n \geq 1$), we have $M(n) = Q_{n,1}$ and thus we *recover* the formula $M(n+1) = M(n) \cdot M_{n+1}$ proven earlier for mixed labyrinth fractals.

A. A. Potapov, W. Zhang, CIE International Conference on Radar (October 2016) : prototypes of ultra-wide band radar antennas based on **labyrinth fractals**





A. A. Potapov, A. A. Potapov Jr., V. A. Potapov, Conference Paper (December 2017) : Fractal radioelements, devices and fractal systems for radar and telecommunications

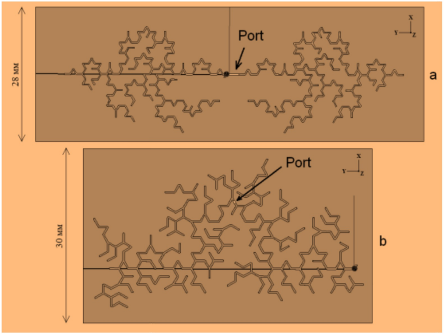


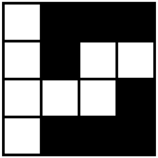
Fig. 5. Results structures for fractal antennas.

T h a n k y o u !

What if the pattern is **not** blocked?

! If only one of the directions of the generating pattern is blocked, then there are pairs of points in the labyrinth fractals such that the length of the arc between them is finite.

Example:



Lemma

The length of the arc in L_∞ between any two distinct points $x, y \in L_\infty$ is **finite** *if and only if the straight line segment from x to y is vertical and is contained in L_∞.*

Wild labyrinth patterns/Wild labyrinth fractals

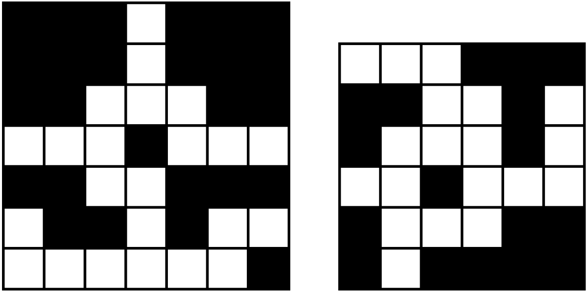
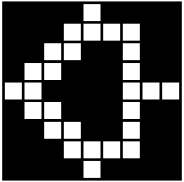


Figure: Examples: wild labyrinth patterns, both *vertically* and *horizontally blocked*

- tree ⟷ connected graph
- uniqueness of v/h exit pair ⟷ existence of v/h exit pair
- corner property

Paths in wild labyrinth fractals. Example

For wild labyrinth fractals the Lemma about the path construction does not hold in general: the squares in the shortest path from the top exit to the bottom exit in $\mathcal{G}(\mathcal{W}_2)$ do not lie whithin the shortest path from the top exit to the bottom exit in $\mathcal{G}(\mathcal{W}_1)$



- in $\mathcal{G}(\mathcal{W}_1)$: $◫_1^{left} = 15$, $◫_1^{right} = 13$,
  $⊟_1 = 15$, $◳_1 = ◲_1 = ◱_1 = ◰_1 = 9$
- the length of the "right" ◫-path in $\mathcal{G}(\mathcal{W}_2)$ is
  $7◫_1 + 2⊟_1 + ◳_1 + ◲_1 + ◱_1 + ◰_1 = 7 \cdot 13 + 2 \cdot 15 + 4 \cdot 9 = 157$
- the length of the "left" ◫-path in $\mathcal{G}(\mathcal{W}_2)$ is
  $3◫_1 + 0 \cdot ⊟_1 + 3◳_1 + 3◲_1 + 3◱_1 + 3◰_1 = 3 \cdot 13 + 4 \cdot 3 \cdot 9 = 147$