Random homogeneous beta-expansions and self-similar measures

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Definition

Let $0 \in S$ be a finite subset of $\mathbb{R}$ and $\beta > 1$. For $x \in \mathbb{R}$, write

$$x = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i} \qquad a_i \in S.$$

We will say that $a_1 a_2 \ldots$ is a **beta-expansion** of x with respect to a digit set S.

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Definition

We define the **support** of a beta-expansion as

$$\{x \mid \ x \text{ has at least one } \beta\text{-expansion}\}$$

Example (Lebesgue measure)

- Let $\beta = 3$ and $S = \{0, 1, 2\}$.
- We see that the beta-expansion of x is the base 3 representation of x.
- The support of this beta-expansion is $[0, 1]$.
- The Hausdorff dimension of the support is 1.
- All $x \in [0, 1]$ have at least 1 expansion, and at most 2 expansions.
- Almost all $x \in [0, 1]$ have a unique expansion.

Example (Cantor Set)

- Let $\beta = 3$ and $S = \{0, 2\}$.
- We see that the beta-expansion of x is the base 3 expansion of x.
- We see that for x in the support, x has a base 3 expansion containing only 0 and 2.
- The support of this beta-expansions is the classical middle third Cantor set.
- The Hausdorff dimension of the support is $\log(2)/\log(3)$.
- All x in the support have a unique expansion.

Example (Bernoulli Convolution)

- Let $\beta = \frac{1+\sqrt{5}}{2}$, the golden ratio and $S = \{0, 1\}$.
- The support of this beta-expansion is $[0, \beta]$.
- The Hausdorff dimension of the support is 1.
- The point $x = 0$ and $x = \beta$ have exactly one beta-expansion.
- The point $x = 1$ has countably many beta-expansions.
- The point $x = \frac{1}{\beta^3 - 1}$ has uncountably many beta-expansions.

Let $G \approx 1.618033$, $q_f \approx 1.754877$, and $q_c \approx 1.787231$.

Theorem (Glendinning, Sidorov, 2001)

- If $q \in (G, q_f]$ then there are exactly 2 non-trival x in the support with unique expansions.
- If $q \in (q_f, q_c)$ then there are countably many x in the support with unique expansions.
- If $q \in [q_c, 2)$ then there are uncountably many x in the support with unique expansions.

Definition

We define a **self-similar measure** on the support of β-expansions as

$$\mu(E) = \mathbb{P}\left\{(a_1 a_2 \ldots) : \sum a_i \beta^{-i} \in E\right\}$$

where E is any Borel set and $\mathbb{P}$ is the product measure on the digit set S.

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- This is a non-atomic self-similar measure.
- For $\beta = 3$ and $S = \{0, 1, 2\}$, This is Lebesgue measure on $[0, 1]$.
- For $\beta = 3$ and $S = \{0, 2\}$, This is the Hausdorff measure on the middle third Cantor set.
- For $\beta = \frac{1+\sqrt{5}}{2}$ and $S = \{0, 1\}$ this is know as the Bernoulli convolution with respect to the golden ratio.

Definition

Given a self-similar measure μ, by the **upper local dimension** of μ at x in the support, we mean the number

$$\overline{\dim}_{loc}\mu(x) = \limsup_{r\to 0^+} \frac{\log \mu([x-r, x+r])}{\log r}.$$

Replacing the lim sup by lim inf gives the **lower local dimension**, denoted $\underline{\dim}_{loc}\mu(x)$. If the limit exists, we call the number the **local dimension** of μ at x and denote this by $\dim_{loc} \mu(x)$.

Example

Let $\beta = 3$ and $S = \{0, 1, 2\}$.
Then for all x in the support,

$$\begin{aligned}\lim_{r\to 0^+} \frac{\log \mu_\beta([x-r, x+r])}{\log r} &= \lim_{n\to\infty} \frac{\log \mu_\beta([x-1/3^n, x+1/3^n])}{\log 1/3^n} \\ &= \lim_{n\to\infty} \frac{\log 2/3^n}{\log 1/3^n} \\ &= 1\end{aligned}$$

Hence all points have local dimension 1.

Example

Let $\beta = 3$ and $S = \{0, 2\}$. Then for all x in the support,

$$\lim_{r \to 0^+} \frac{\log \mu_\beta([x - r, x + r])}{\log r} = \lim_{n \to \infty} \frac{\log \mu_\beta([x - 1/3^n, x + 1/3^n])}{\log 1/3^n}$$
$$= \lim_{n \to \infty} \frac{\log 1/2^n}{\log 1/3^n}$$
$$= \frac{\log 2}{\log 3} \approx 0.6309$$

Hence all points have local dimension $\log 2 / \log 3$.

Example

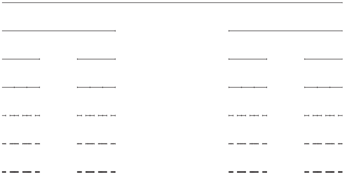
Let $\beta \approx 1.618$ be the golden ratio and $S = \{0, 1\}$.

- **Feng - 2005, 2009:** The set of attainable local dimensions of μ is the interval, $\approx [0.9404, 1.4404]$.
- **Alexander, Zagier - 1991:** Most points have local dimension ≈ 0.9957.

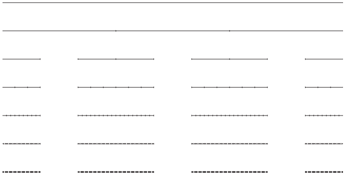
Example

Let $\beta = 3$. Let $S_1 = \{0, 1, 2\}$ and $S_2 = \{0, 2\}$. Pick a random fixed sequence $\mathbf{b} = \{b_i\}_{i=1}^{\infty} \in \{1, 2\}^{\mathbb{N}}$. Consider the set of all β-expansions

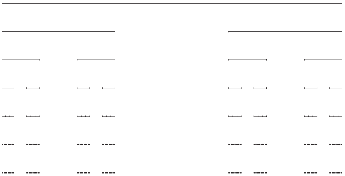
$$\left\{ x \mid x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, a_i \in S_{b_i} \right\}$$



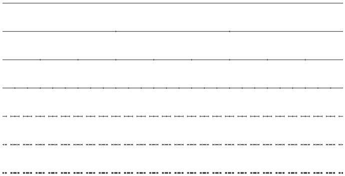
(a) $\mathbf{b} = 2, 2, 1, 2, 1, \ldots$



(b) $\mathbf{b} = 1, 2, 2, 2, 1, \ldots$



(c) $\mathbf{b} = 2, 2, 2, 1, 1, \ldots$



(d) $\mathbf{b} = 1, 1, 1, 2, 1, \ldots$

- Assuming the b_i are choosen randomly, and with equal probability, then the Hausdorff dimension of the support, almost surely, is $\frac{\log(6)}{\log(9)}$.
- Let μ_H be the Hausdorff measure with respect to this dimension. Then the μ_H measure of this support, almost surely, is 0.

Definition

- We say that a iterated function system satisfies the **Strong Separation Condition** if images under the maps never overlap.
- We say that a iterated function system satisfies the **Open Set Condition** if images under the maps never overlap non-trivially.
- We say that a iterated function system satisfies the **Finite Type Condition** if images under the maps overlap in only a finite number of ways (up to rescaling)

- **Falconer 1986, Graf 1987:** Determined their almost sure Hausdorff dimension and measure properties.
- As before, we can associate a measure μ with respect to **b**. These measures have a number of interesting almost sure results.
- **Olsen 1994, Arbeiter Patzschke 1996:** Considered the multifractal spectrum under the random analog of the open set condition

Theorem (Hare, H., Troscheit, to appear)

Consider a family of iterated function systems that satisfy the uniform strong separation conditions. Let $\mu_{\mathbf{b}}$ *be the measure constructed by this family for particular* $\mathbf{b}$*. There exists an* $\underline{\alpha}$ *and* $\overline{\alpha}$ *such that for almost all* $\mathbf{b}$ *we have*

$$\begin{aligned}[\underline{\alpha}, \overline{\alpha}] &= \{\dim_{loc} \mu_{\mathbf{b}}(x) \ : \ x \in Support\} \\ &= \{\underline{\dim}_{loc}\mu_{\mathbf{b}}(x) \ : \ x \in Support\} \\ &= \{\overline{\dim}_{loc}\mu_{\mathbf{b}}(x) \ : \ x \in Support\}.\end{aligned}$$

- Let $v_1, v_2, v_3, \ldots, v_m$ be the finite number of ways that images of this map can overlap.
- We construct the adjacency matrix $A_{\mathbf{b}_i}$ which determines the number of v_i at level $n+1$ from the number of v_i at level n where the digit choice is taken from $S_{\mathbf{b}_n}$.
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Proposition (Hare, H., Troscheit, to appear)

Let the system of iterated function systems satisfy the finite type condition. For almost all $\mathbf{b}$, the box-counting dimension and Hausdorff dimension of the support is equal to

$$\lim_{n\to\infty} \frac{\log \left\| A_{\mathbf{b}_1} \ldots A_{\mathbf{b}_n} \right\|}{n |\log r|}.$$

Fact

We can compute the local dimension of a point x with respect to $\mu_{\mathbf{b}}$ be a matrix product $\|B_1 B_2 B_3 \ldots B_n\|^{1/n}$ where the B_n depends upon

- *Which v_n we are in at each level*
- *Which digit set $S_{\mathbf{b}_n}$*
- *The location of x within v_n at level n.*

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Definition

We define $\mathbf{B}_j$ as the value of $B_{k_j+1} \ldots B_{k_{j+1}}$ and N_j the length of this product.

Theorem (Hare, H., Troscheit, to appear)

Consider a random iterated function system of finite type and assume that the associated random self-similar measures, $\mu_{\mathbf{b}}$, are regular. If the RIFS is commuting, then for almost all **b***, the set of attainable local dimensions for the measure $\mu_{\mathbf{b}}$ is the closed interval,*

$$\left[\frac{\mathbb{E}\left(\log \overline{\mathbf{B}}_1\right)}{\mathbb{E}(N_1)\log r}, \frac{\mathbb{E}\left(\log \underline{\mathbf{B}}_1\right)}{\mathbb{E}(N_1)\log r}\right].$$

Thank you