Totally positive quadratic integers and numeration

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Siegel 1945 $K=\mathbb{Q}(\sqrt{5})$. Then $\{x^2+y^2+z^2 : x,y,z\in\mathcal{O}_K\} = \mathcal{O}_K^+\cup\{0\}$
$=\{\alpha\in\mathcal{O}_K : \alpha\geq 0, \alpha'\geq 0\}$. Doable only for K and $\mathbb{Q}$.

Notation
- D square free, e.g. $D=174$
- $\mathcal{O}_K=\mathbb{Z}[\sqrt{D}]$ $\quad D\not\equiv 1 \pmod 4$ (everything works the same there but is more complicated)
- $\alpha\succ 0$ iff $\alpha>0$ & $\alpha'>0$, $\mathcal{O}_K^+=\{\alpha\in\mathcal{O}_K : \alpha\succ 0\}$.
- $\alpha = x+y\sqrt{D}$, $x>0$, $-\frac{x}{\sqrt{D}}<y<\frac{x}{\sqrt{D}} \Rightarrow \alpha\in\mathcal{O}_K^+$

Indecomposables $\alpha\succ 0$, $\alpha\neq\beta+\gamma$, $\beta,\gamma\succ 0$

$$\sqrt{174}=[13,\overline{5,4,5,26}]=[u_0,u_1,u_2,\dots]$$

convergents $\frac{1}{0}, \frac{13}{1}, \frac{66}{5}, \frac{277}{21}, \dots$

$\alpha_{-1}=1+0\sqrt{D}$, $\alpha_0=13+1\sqrt{D}$, $\alpha_1=66+5\sqrt{D}, \dots$

indecomposables: $\alpha_i + r\alpha_{i+1}$, $i\geq -1$ odd, $0\leq r\leq u_{i+2}-1$
+ Galois conjugates

simplified notation:

$$\cdots < \alpha_{-1}'+\alpha_0' < \alpha_{-1}=1 < \alpha_{-1}+\alpha_0 < \alpha_{-1}+2\alpha_0 < \alpha_{-1}+3\alpha_0 < \alpha_1 < \cdots$$
$$\cdots < \beta_{-1}=\beta_1' < \beta_0 < \beta_1 < \beta_2 < \cdots$$

Presentation

$$\beta_{j+1}=v_j\beta_j-\beta_{j-1}, \quad v_j=\begin{cases} 2u_0+2, & \beta_j=1 \\ u_{i+1}+2, & \beta_j=\alpha_i \text{ or } \alpha_i' \\ 2, & \text{otherwise}\end{cases} \qquad (*)$$

Thm $\mathcal{O}_K^+=\langle \beta_j \mid (*)\rangle$

Greedy $\alpha \succ 0$, $\alpha = \sum k_j \beta_j$

$\ldots k_3 k_2 k_1 k_0 . k_{-1} k_{-2} \ldots$ lexicographically maximal.

- greedy algorithm
  - j maximal s.t. $\beta_j \prec \alpha$
  - k_j maximal s.t. $k_j \beta_j \prec \alpha$
  - $x \mapsto x - k_j \beta_j$, $j \mapsto j-1$

Thm • Greedy alg. ~~gives~~ terminates in finitely many steps and gives the greedy representation.

- $(k_j)_{j \in \mathbb{Z}}$ is greedy of some $\alpha \succ 0$ if and only if:
  1) finitely many k_j non-zero
  2) $0 \leq k_j \leq v_j - 1$
  3) $(v_j - 1, v_{j+1} - 2, v_{j+2} - 2, \ldots, -2, \ldots, -2, v_j - 1)$ does not occur.
- $(k_j)_{j \in \mathbb{Z}}$ greedy of $\alpha \Leftrightarrow (k_j')_{j \in \mathbb{Z}}$ greedy of α'
- $\alpha, \beta \succ 0$. Then $\alpha < \beta \Leftrightarrow$ greedy of $\alpha <_{lex}$ greedy of β

Application • $S := \{\alpha \succ 0 : \forall \beta \in \mathcal{O}_K \setminus \{0\} : \alpha \prec \beta^2\}$

Prop $\sqrt{D} = [t, \overline{5, 4, 5, 2t}]$ squarefree, $D \not\equiv 1 \bmod 4$.

Then $\#(S/U_K^2) = \dfrac{387t - 26}{55}$

- We can do this (on PC) for any "family" of D's.
- Useful in the study of universal quadratic forms.