Characterization of rational matrices that admit digit systems with finiteness property

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Sources

- J. J. & J .M. Thuswaldner, *Characterization of rational matrices that admit digit systems with the finiteness property*, arXiv: https://arxiv.org/abs/1801.01839
- More recent (yet unpublished) results from the work that is currently in progress.

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N. S. in orders (from prof. A. Pethő talk)

- $\alpha \in \mathbb{C}$ - algebraic number, and $\mathcal{D} = \{d_1, d_2, \dots, d_k\} \subset \mathbb{Z}$
- **Number system:**

$$\mathcal{D}[\alpha] = \{\epsilon_0 + \alpha\epsilon_1 + \cdots + \epsilon_{l-1}\alpha^{l-1}\}.$$

- **Uniqueness:** Each $\beta \in \mathcal{D}[\alpha]$, has **unique expression**.
- **Finiteness:** $\mathcal{D}[\alpha] = \mathbb{Z}[\alpha]$.
- **Standard N.S.:** = **Uniqueness** + **Finiteness**
- (Sometimes, one also requires $0 \in \mathcal{D}$ for S.N.S).

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Examples of N. S. with/without (U) or (F)

- $(2, \{0, 1\})$ – d.s. in $\mathbb{Z}$, with (U) but not (F).
- $(2, \{-1, 0, 1\})$ – d.s. in $\mathbb{Z}$ with (F), but no (U), $1 = \overline{-11}$.
- $(2, \{-1, 0\})$ – again, we loose (F): $1 = \overline{(-1)^\infty}$.
- There **exist no base–2 S.N.S. with both** (U+F) in $\mathbb{Z}$!
- $(3, \{-1, 0, 1\})$ – standard d.s. in $\mathbb{Z}$, with (F) and (U).

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Relevant work on the N.S. with Finiteness property

[1.] C. Frougny, *Representation of numbers and finite automata*, Math. Systems Theory **25** (1992).

[2.] S. Akiyama, P. Drungilas, J. Jankauskas, *Height reducing problem on algebraic integers*, Funct. Approx. Comment. Math. **26** (1) (2012), 105–119.

[3.] S. Akiyama, T. Zaïmi, *Comments on the height reducing property I*, Cent. Eur. J. Math. **11** (9) (2013), 1616–1627.

[4.] S. Akiyama, J. M. Thuswaldner, T. Zaïmi, *Characterization of numbers that satisfy the height reducing porperty*, Indag. Math. **26** (1) (2015), 24–27.

[5.] S. Akiyama, J. M. Thuswaldner, T. Zaïmi, *Comments on the height reducing property II*, Indag. Math. **26** (1) (2015), 28–39.

Key results on the Finiteness property in N. S.

- Theorem 1 (Frougny, 1992)
  If α *is an algebraic number with no algebraic conjugate* $|\alpha'| = 1$ *over* $\mathbb{Q}$ *, then, for each* $\beta \in \mathbb{Z}[\alpha]$*, one can decide whether* $\beta \in \mathcal{D}[\alpha]$ *with a finite automaton.*
- Theorem 2 (Akiyama, Thuswaldner, Zaïmi, 2015)
  The finiteness property $\mathbb{Z}[\alpha] = \mathcal{D}[\alpha]$ *holds for some finite* $\mathcal{D} \subset \mathbb{Z}$ *precisely when* α *is an algebraic number with no algebraic conjugate over* $\mathbb{Q}$ *of absolute value* $|\alpha'| < 1$.

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Digit systems $(A, \mathcal{D})$ in lattices

- A. Vince, *Replicating tessellations*, SIAM J. Discrete Math. **6** (3) (1993).
- **Base**: $A \in M_n(\mathbb{Z})$ – $n \times n$ integer matrix.
- **Digit set**: $\mathcal{D} = \{\mathbf{d}_1, \ldots, \mathbf{d}_k\} \subset \mathbb{Z}^n$
- **Digit system:** the pair $(A, \mathcal{D})$ with the set of possible radix expansions of finite length:

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Facts about integral $(A, \mathcal{D})$ in lattices $\mathbb{Z}^d$.

- **Necessary conditions** for standard d. s.
  - A must be **expanding**: $\forall$ eigenvalue $|\lambda| > 1$;
  - $\mathbb{Z}^n / A\mathbb{Z}^n = \mathcal{D}$.
- If $A \in M_n(\mathbb{Z}^n)$ is expanding, there always exists some finite set $\mathcal{D} \in \mathbb{Z}^n$, s.t. $(A, \mathcal{D})$ has property (F).
- The mapping $\Phi : \mathbb{Z}^n \mapsto \mathbb{Z}^n$

$$\Phi(\mathbf{x}) = A^{-1}(\mathbf{x} - \mathbf{r}(\mathbf{x})),$$

with the remainder

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- The mapping $\Phi : \mathbb{Z}^n \mapsto \mathbb{Z}^n$

$$\Phi(\mathbf{x}) = A^{-1}(\mathbf{x} - \mathbf{r}(\mathbf{x})),$$

  with the remainder

$$\mathbf{r} : \mathbb{Z}^n \mapsto \mathcal{D}, \mathbf{r}(\mathbf{x}) \equiv \mathbf{x} \pmod{A\mathbb{Z}^n}.$$

  has finite attractor set in $\mathbb{Z}^n$.

Example of standard D.S. in $\mathbb{Z}^2$

- Standard d.s. in $\mathbb{Z}^2$:

$$A = \begin{pmatrix} 0 & -2 \\ 1 & -2 \end{pmatrix}, \quad \mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

- $(A, \mathcal{D})$ generalize n.s. $(\alpha, \mathcal{E})$ in orders (take A to be companion matrix of α with $\mathcal{D} = \{\varepsilon \mathbf{e}_1, \varepsilon \in \mathcal{E}\}$
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Generalizing finiteness result for $(A, \mathcal{D})$.

- We allow $A \in M_n(\mathbb{Q})$, also with eigenvalues $|\lambda| = 1$.
- Minimal A-invariant $\mathbb{Z}$-module that contains $\mathbb{Z}^n$:

$$\mathbb{Z}^n[A] := \bigcup_{k=1}^{\infty} \left(\mathbb{Z}^n + A\mathbb{Z}^n + \cdots + A^{k-1}\mathbb{Z}^n \right).$$

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  Let A be an $n \times n$ matrix with rational entries. There is a digit set $\mathcal{D} \subset \mathbb{Z}^n[A]$ that makes $(A, \mathcal{D})$ a digit system in $\mathbb{Z}^n[A]$ with finiteness property iff A has no eigenvalue λ with $|\lambda| < 1$. The digit set $\mathcal{D}$ can even be chosen to be a subset of $\mathbb{Z}^n$.

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Main difficulties

- When $A \notin M_n(\mathbb{Z})$, usually $\mathbb{Z}^n[A] \subset \mathbb{R}^n$ **no longer** a lattice, for instance, $\mathbb{Z}[3/2]$.
- When $|\lambda| = 1$, A^{-1} **not contractive**; if λ not a root of unity, infinite orbits $\{A^{-n}\mathbf{x}, \mathbf{x} \in \mathbb{R}^n\}$ show up.
- When A contains Jordan blocks $J(\lambda)$ of order $\geqslant 2$, $|\lambda| = 1$, **unbounded orbits** show up, i.e.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, A^{-n}\mathbf{x} = \begin{pmatrix} x_1 - nx_2 \\ x_2 \end{pmatrix}$$

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Sketch of the Proof Th.2 $\to$ Th.3

Step 1. Let $P \in \mathbb{Z}[x]$ be the minimal polynomial of α. If α has no conjugate $|\alpha'| < 1$, Then $\mathbb{Z}[\alpha] = \mathcal{N}[\alpha]$ with a finite $\mathcal{N} \in \mathbb{Z}$. Since $\mathbb{Z}[\alpha] \cong \mathbb{Z}[x]/(P)$.

$$\mathbb{Z}[x] = \mathcal{N}[x] + (P), \text{ where}$$

$$\mathcal{N}[x] = \{b_0 + b_1 x + \cdots + b_k x^k : b_j \in \mathcal{N}, 0 \leqslant j \leqslant k\}.$$

Step 2. If $P, Q \in \mathbb{Z}[x]$ satisfy

$$\mathbb{Z}[x] = \mathcal{L}[x] + (P), \quad \mathbb{Z}[x] = \mathcal{M}[x] + (Q),$$

for $\mathcal{M}$, $\mathcal{L} \subset \mathbb{Z}$ finite, then there exists a finite $\mathcal{N} \in \mathbb{Z}$, such that $\mathbb{Z}[x] = \mathcal{L}[x] + (PQ)$. For $S \in \mathbb{Z}[x]$,

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Step 3. Let $P \in \mathbb{Z}[x]$, $m \in \mathbb{N}$, satisfy $\mathbb{Z}[x] = \mathcal{N}[x] + (P^m)$, with $\mathcal{N} \subset \mathbb{Z}$; set $d := m \cdot \deg P$ and consider

$$C = C(P^m) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -a_0/a_d \\ 1 & 0 & 0 & \dots & 0 & -a_1/a_d \\ 0 & 1 & 0 & \dots & 0 & -a_2/a_d \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -a_{d-2}/a_d \\ 0 & 0 & 0 & \dots & 1 & -a_{d-1}/a_d \end{pmatrix}$$

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D.S. with (F) for Rotation Matrices

- For practical computation of D.S. with (F), one needs effective version of Th.2 by Akiyama, Thuswaldner and Zaïmi.
- A real matrix $A \in M_n(\mathbb{Q})$ is called **general rotation**, if it is fully diagonalizable over $\mathbb{C}$ and all eigenvalues $\lambda \in \mathbb{C}$ of A are of absolute value $|\lambda| = 1$.
- If A is a general rotation, $\exists Q \in M_n(\mathbb{R})$, such that $Q^{-1}AQ$ takes block–diagonal form with blocks

$$(\pm 1), \qquad \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

- A^{-1}–invariant norm on R^n is defined by

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$$||\mathbf{x}||_{A^{-1}} = \left|\left|Q^{-1}\mathbf{x}\right|\right|_{\text{Euclidean}}, \qquad \left|\left|A^{-1}\mathbf{x}\right|\right|_{A^{-1}} = ||\mathbf{x}||_{A^{-1}}.$$

Digit sets with *good* convex enclosure

- A digit set $\mathcal{D} \subset \mathbb{Z}^n[A]$ is said to have **a good enclosure**, when for each residue $\mathbf{r} \in \mathbb{Z}^n[A]/A\mathbb{Z}^n[A]$, the interior of the convex hull of digits

$$\mathcal{D}(\mathbf{r}) := \{\mathbf{d} \in \mathcal{D} : \mathbf{d} \equiv \mathbf{r} \pmod{A\mathbb{Z}^n[A]}\}$$

viewed as points in $\mathbb{R}^n$, contains the origin $\mathbf{0}_n$.

- **Division function** $\Phi : \mathbb{Z}^n[A] \mapsto \mathbb{Z}^n[A]$ by

$$\Phi(\mathbf{x}) := A^{-1}(\mathbf{x} - \mathbf{r}(\mathbf{x})).$$

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$$\mathbf{r}(\mathbf{x}) = \begin{cases} \mathbf{r}(\mathbf{x}) \equiv \mathbf{x} \pmod{A\mathbb{Z}^n[A]}; \\ \text{minimizes } ||\mathbf{x} - \mathbf{r}(\mathbf{x})||_{A^{-1}} \text{ in } \mathcal{D}; \end{cases}$$

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New result (as of yet unpublished)

- Theorem 4 (J.J. & J. Thuswaldner)
  Let $A \in \mathbb{Q}^{n \times n}$ be generalized rotation and suppose that $\mathcal{D} \subset \mathbb{Z}^n$ contains $\mathbb{Z}^n/(A\mathbb{Z}^n \cap \mathbb{Z}^n)$. If $\mathcal{D}$ is arithmetically convex, then the division mapping $\Phi : \mathbb{Z}^n[A] \mapsto \mathbb{Z}^n[A]$ has finite attractor set $\mathcal{A}_\Phi \subset \mathbb{Z}^n$. In particular, Φ is ultimately periodic function with a finite number of possible smallest periods all of which lie in $\mathbb{Z}^n$.
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Example (beginning)

- The 2×2 matrix

$$A = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}, \qquad (3/5)^2 + (4/5)^2 = 1.$$

- This is a rotation by the angle

$$\theta = 53.1301\ldots^\circ$$

  that **is not a rational multiple of** 2π.
- Since A is orthogonal, $||\mathbf{x}||_{A^{-1}} = ||\mathbf{x}||_{\text{Euclidean}}$, $Q = \mathbf{1}_2$.

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Example (continued) – convex enclosure of $\mathbf{0}_2$

- The set $\mathcal{L} = \mathbb{Z}^2 \cap A\mathbb{Z}^2$ is a lattice with the basis matrix $\mathcal{L} = T\mathbb{Z}^2$:
$$T := \begin{pmatrix} 7 & 5 \\ 1 & 0 \end{pmatrix}.$$
- The residue set

$$\mathcal{R} := \mathbb{Z}^2/L\mathbb{Z}^2 = \left\{ \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\} =$$
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Since the residue group $\mathbb{Z}^2/\mathcal{L}\mathbb{Z}^2$ is of prime order 5, this is also a full residue group for $\mathbb{Z}^2[A]/A\mathbb{Z}^2[A]$.

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Example (continued): convex enclosures

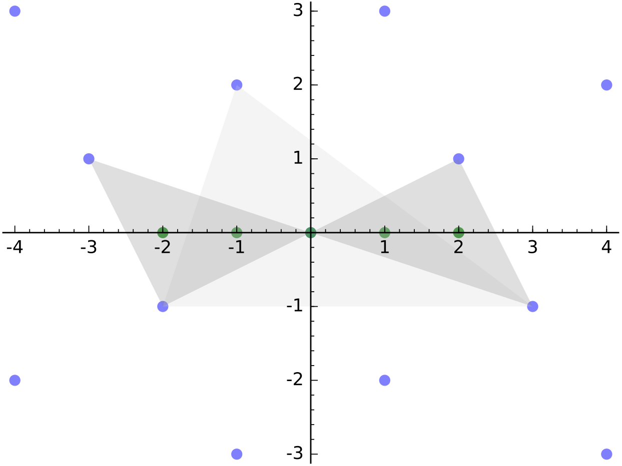

Figure: Lattice $\mathcal{L} = \mathbb{Z}^2 \cap A\mathbb{Z}^2$ (blue), residue set $\mathcal{R} = \mathbb{Z}^2/\mathcal{L}$ (green). Triangles (grey and light grey) with vertices in $\mathcal{L}$ that enclose $\mathcal{R}$ with vertices $\mathcal{T}_1, -\mathcal{T}_1, \mathcal{T}_2 \subset \mathcal{L}$.

Example (continued): initial digit set

- $\mathcal{D}(\mathbf{0}_2) := \mathbf{0}_2 - \mathcal{T}_2 = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}$
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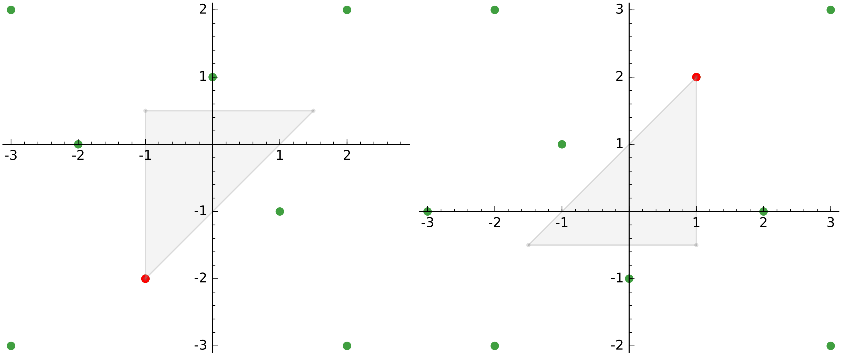
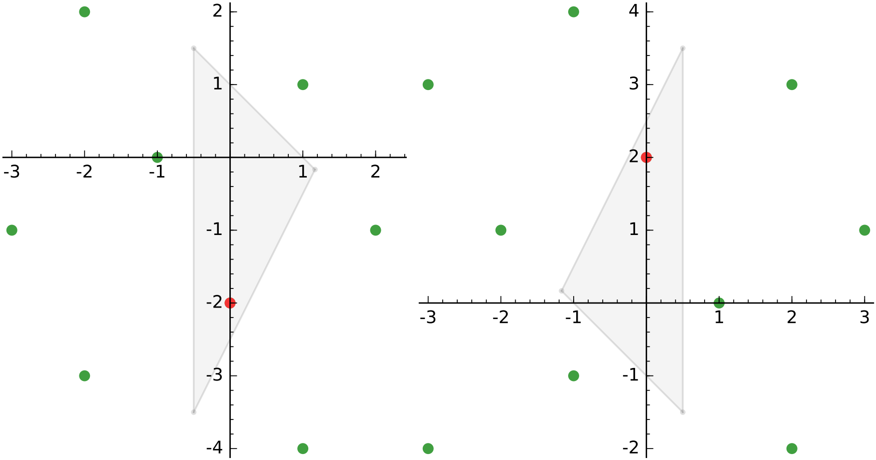
Example (continued): attractor set in $\mathbb{Z}^2$, I



(a) $\mathbf{r} = (-2, 0)^T$ (b) $\mathbf{r} = (2, 0)^T$

Figure: Attractor points $\mathrm{Attr}_\Phi(\mathbf{r})$ (red) for each residue class $\mathbf{r} \in \mathbb{Z}^2/\mathcal{L}$. Green colored are points from $\mathbf{r} + \mathcal{L}$.

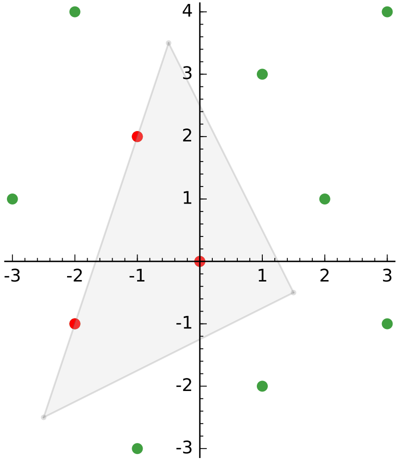
Example (continued): attractor set, II



(a) $\mathbf{r} = (-1, 0)^T$ (b) $\mathbf{r} = (1, 0)^T$

Figure: Attractor points $\mathrm{Attr}_\Phi(\mathbf{r})$ (red) for each $\mathbf{r} \in \mathbb{Z}^2/\mathcal{L}$. Green colored are points from $\mathbf{r} + \mathcal{L}$.

Example (continued): attractor set, III



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- The final digit set

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- $\forall \mathbf{x} \in \mathbb{Z}^n[A]$, $\Phi^n(\mathbf{x})$ always visits

$$\mathcal{P} = \left\{ \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-1\\-2\end{pmatrix} \right\}.$$

- The final digit set

$$\mathcal{D} = \mathcal{D}' \cup \mathcal{P},$$

Example (continued): orbits of attractor points

- $\Phi\begin{pmatrix}0\\0\end{pmatrix} = \Phi\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$,
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Example (end)

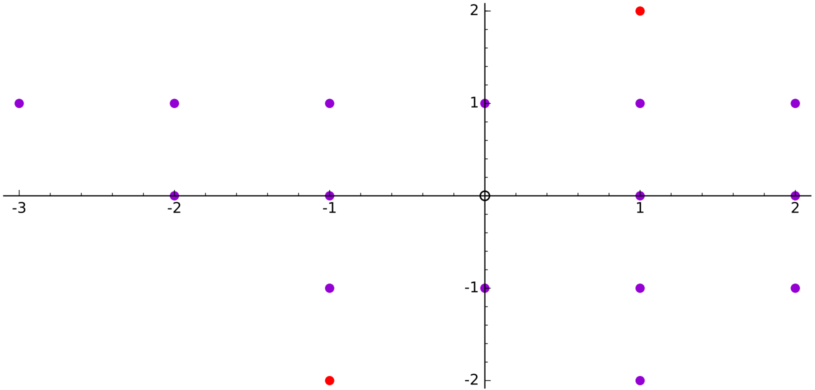

Figure: Set $\mathcal{D}$: Pre-periodic digits (violet) and periodic digits (red); Note that $\mathbf{0}_2 \notin \mathcal{D}$ and $\#\mathcal{D} = 17$.

Open problems

Problem 1

Suppose an $n \times n$ matrix $A \in M_n(\mathbb{Q})$ with rational entries has eigenvalues λ with $|\lambda| \geq 1$, and that at least one eigenvalue is of absolute value $|\lambda| = 1$. Is it true that a digit system $(A, \mathcal{D})$ in $\mathbb{Z}^n[A]$ that has finiteness property does not admit unique representation property?

Problem 2

Suppose that A satisfies the assumptions of P1, and that $(A, \mathcal{D})$ in $\mathbb{Z}^n[A]$ has the finiteness property. What is the smallest possible size $\#\mathcal{D}$ of the digit set?

Problem 3

Suppose that A satisfies the assumptions of P1, and let $(A, \mathcal{D})$ be arbitrary digit system in $\mathbb{Z}^n[A]$ (not necessarily having a finiteness property). Is there a way to check whether $\mathbf{z} \in \mathbb{Z}^n[A]$ in $(A, \mathcal{D})$ (by finite automaton or otherwise)?

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Suppose that A *satisfies the assumptions of P1, and that* $(A, \mathcal{D})$ *in* $\mathbb{Z}^n[A]$ *has the finiteness property. What is the smallest possible size* $\#\mathcal{D}$ *of the digit set?*

Problem 3

Suppose that A *satisfies the assumptions of P1, and let* $(A, \mathcal{D})$ *be arbitrary digit system in* $\mathbb{Z}^n[A]$ *(not necessarily having a finiteness property). Is there a way to check whether* $\mathbf{z} \in \mathbb{Z}^n[A]$ *in* $(A, \mathcal{D})$ *(by finite automaton or otherwise)?*

The End

Thank you!

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