Unique expansions on fat Sierpinski gaskets

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Given $\beta \in (1,2)$, the fat Sierpinski gasket S_β is the self-similar set generated by the iterated function system

$$f_A(x,y) = \left(\frac{x}{\beta}, \frac{y}{\beta}\right), \quad f_B(x,y) = \left(\frac{x+1}{\beta}, \frac{y}{\beta}\right),$$
$$f_C(x,y) = \left(\frac{x}{\beta}, \frac{y+1}{\beta}\right).$$

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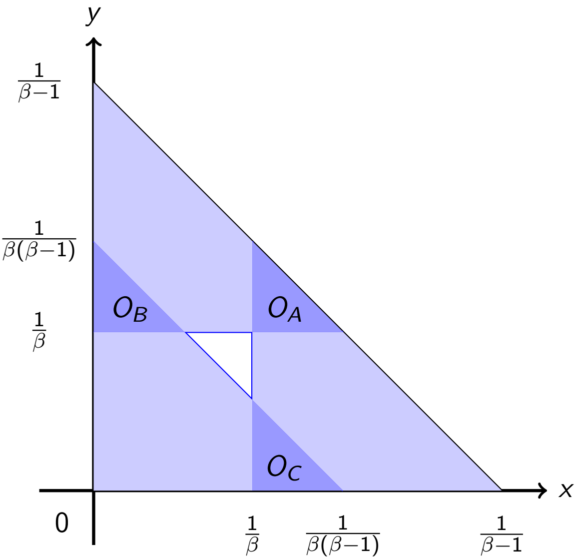
Then

$$S_\beta = \left\{ \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} : d_i \in \{A, B, C\} \ \forall i \geq 1 \right\},$$

where

$$A = (0,0), \quad B = (1,0), \quad C = (0,1).$$



y
$\frac{1}{\beta-1}$
$\frac{1}{\beta(\beta-1)}$
$\frac{1}{\beta}$
O_B
O_A
O_C
x
0
$\frac{1}{\beta}$
$\frac{1}{\beta(\beta-1)}$
$\frac{1}{\beta-1}$

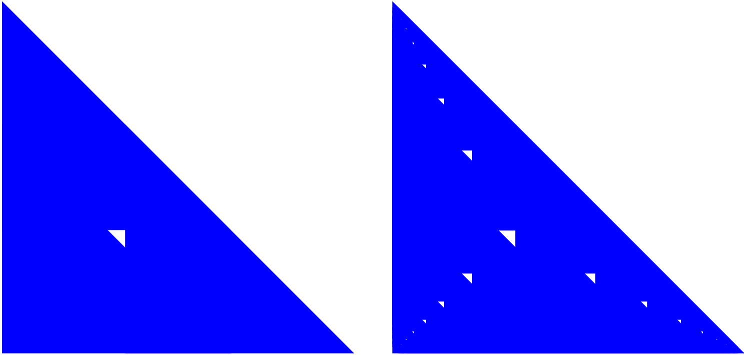


Figure: S_β with $\beta = 1.54$.

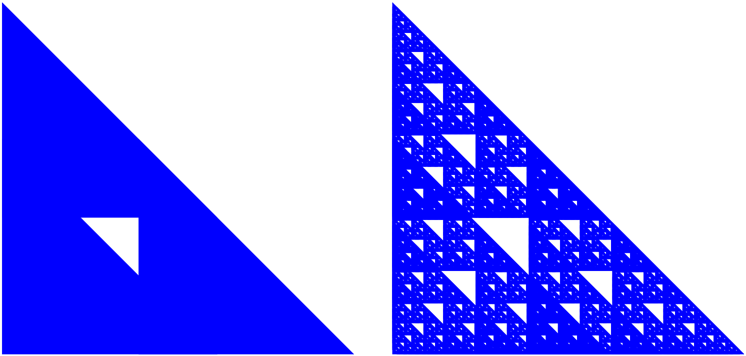


Figure: S_β with $\beta = 18/11 \approx 1.63636$.

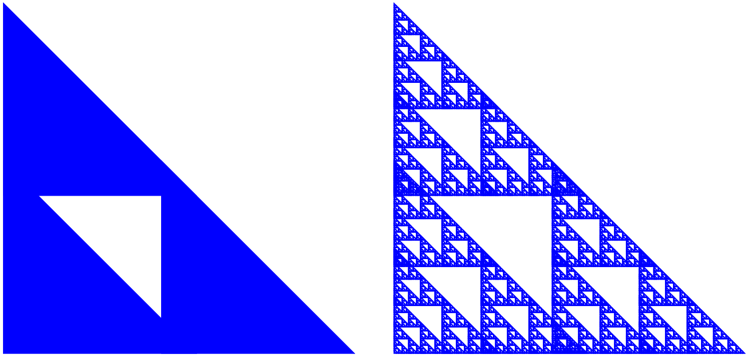


Figure: S_β with $\beta = 20/11 \approx 1.81818$.

- If $\beta \leq 1.5$, then $S_\beta = \Delta$ is a triangle.

- If $\beta \in (1.5, 1.54369]$, then S_β contains interior points (Broomhead, Montaldi and Sidorov 2004).

This was extended later by Hasselblatt and Plante (2014) to $[1.54512, 1.5456] \cup [1.54655, 1.54847] \cup [1.55255, 1.55304]$.

...

Let

$$U_\beta := \left\{ \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \in S_\beta : \sum_{i=1}^{\infty} \frac{d_{n+i}}{\beta^i} \notin O_\beta \;\forall n \geq 0 \right\},$$

where $O_\beta := O_A \cup O_B \cup O_C$. Then every point in U_β has a unique β-expansion with alphabet $\{A, B, C\} = \{(0,0), (1,0), (0,1)\}$.

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Accordingly, let

$$\mathbf{U}_\beta := \left\{ (d_i) \in \{A, B, C\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \in U_\beta \right\}.$$

Proposition

Let $\beta \in (1,2)$. *Then* $(d_i) \in \mathbf{U}_\beta$ *if and only if* $(d_i) \in \{A, B, C\}^{\mathbb{N}}$ *satisfying*

$$\begin{cases} d^1_{n+1} d^1_{n+2} \ldots \prec \delta(\beta) & \textit{if} \quad d^1_n = 0, \\ d^2_{n+1} d^2_{n+2} \ldots \prec \delta(\beta) & \textit{if} \quad d^2_n = 0, \\ d^{\oplus}_{n+1} d^{\oplus}_{n+2} \cdots \succ \overline{\delta(\beta)} & \textit{if} \quad d^{\oplus}_n = 1. \end{cases}$$

- Here $d_n = (d^1_n, d^2_n) \in \{A, B, C\}$ and $d^{\oplus}_n := d^1_n + d^2_n$.
- $\delta(\beta)$ is the quasi-greedy β-expansion of 1, and $\overline{\delta(\beta)} = (1 - \delta_i(\beta))$.

Proposition

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- Here $d_n = (d^1_n, d^2_n) \in \{A, B, C\}$ and $d^{\oplus}_n := d^1_n + d^2_n$.
- $\delta(\beta)$ is the quasi-greedy β-expansion of 1, and $\overline{\delta(\beta)} = (1 - \delta_i(\beta))$.
- $\sigma(\mathbf{U}_\beta) \subset \mathbf{U}_\beta$, and the set-valued map $\beta \mapsto \mathbf{U}_\beta$ is non-decreasing.

Theorem (Sidorov, 2007)

Let $\beta_G \approx 1.46557$ *be the root of* $x^3 - x^2 - 1 = 0$, *i.e.,* $\delta(\beta_G) = (100)^\infty$. *Then*

$$\mathbf{U}_\beta = \{A^\infty, B^\infty, C^\infty\} \quad \Leftrightarrow \quad 1 < \beta \le \beta_G.$$

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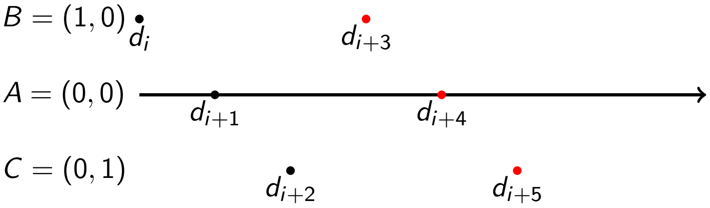

Figure: The presentation of the block $d_i \dots d_{i+5} = BACBAC$.

Example

Let $\beta_2 \approx 1.5385$ such that $\delta(\beta_2) = (101000)^\infty$. Then for any $\beta \in (\beta_G, \beta_2]$ we have

$$\mathbf{U}_\beta \setminus \mathbf{U}_{\beta_G} = \{*(BAC)^\infty, *(ABC)^\infty\}.$$

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Example

Let $\beta_3 \approx 1.55263$ such that $\delta(\beta_3) = (101001000100)^\infty$. Then for any $\beta \in (\beta_2, \beta_3]$ we have

$$\mathbf{U}_\beta \setminus \mathbf{U}_{\beta_2} = \{*(BABCAC)^\infty, *(CBCABA)^\infty, *(ACABCB)^\infty\}.$$

$$\delta(\beta_1) = (100)^\infty, \qquad \mathbf{U}_{\beta_1} = \{A^\infty, B^\infty, C^\infty\};$$

$$\delta(\beta_2) = (101\,000)^\infty, \quad \mathbf{U}_{\beta_2} \setminus \mathbf{U}_{\beta_1} = \{*(BAC)^\infty, *(CBA)^\infty, *(ACB)^\infty\};$$

$$\delta(\beta_3) = (101001\,000100)^\infty,$$

$$\mathbf{U}_{\beta_3} \setminus \mathbf{U}_{\beta_2} = \{*(BABCAC)^\infty, *(CBCABA)^\infty, *(ACABCB)^\infty\}.$$

What is next?

$$\delta(\beta_1) = (100)^\infty, \qquad \mathbf{U}_{\beta_1} = \{A^\infty, B^\infty, C^\infty\};$$

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$$\mathbf{U}_{\beta_3} \setminus \mathbf{U}_{\beta_2} = \{*(BABCAC)^\infty, *(CBCABA)^\infty, *(ACABCB)^\infty\}.$$

What is next?

$$\delta(\beta_4) = (101001000101\,000100101000)^\infty.$$

$$\mathbf{U}_{\beta_4} \setminus \mathbf{U}_{\beta_3} = \ldots?$$

$$\cdots$$

Let $\Omega := \{000, 001, 100, 101\}$. Define

$$\Theta : \Omega \to \Omega; \quad 000 \mapsto 101,\ 100 \mapsto 001,\ 001 \mapsto 100,\ 101 \mapsto 000.$$

This induces the map $\phi : \Omega^* \to \Omega^*; \quad \mathbf{t} \mapsto \mathbf{t}^+\Theta(\mathbf{t}^+)$.

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Definition

For $n \geq 1$ let β_n and β_c be defined by

$$\delta(\beta_n) = (\phi^{n-1}(100))^\infty, \quad \text{and} \quad \delta(\beta_c) = \lim_{n\to\infty} \phi^n(100).$$

Then $\beta_1 = \beta_G < \beta_2 < \cdots$ and $\beta_n \nearrow \beta_c$ as $n \to \infty$.

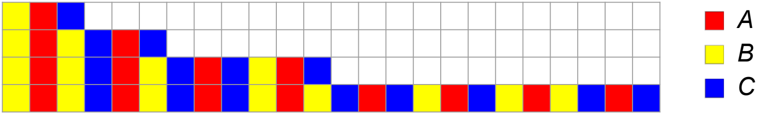
Type-A Thue-Morse words.

$$\Phi_A : \{A, B, C\} \to \{A, B, C\}; \quad A \mapsto A,\ B \mapsto C,\ C \mapsto B.$$

Set $\mathbf{u}_0 = A$, $\mathbf{u}_1 = BAC$, and for $n \geq 1$ we set $\mathbf{u}_{n+1} := \mathbf{u}_n^B \Phi_A(\mathbf{u}_n^B)$. Then

$$\mathbf{u}_2 = BABCAC, \quad \mathbf{u}_3 = BABCAB\,CACBAC,$$

$$\mathbf{u}_4 = BABCAB\;CACBAB\;CACBACBABCAC.$$



Type-B Thue-Morse words.

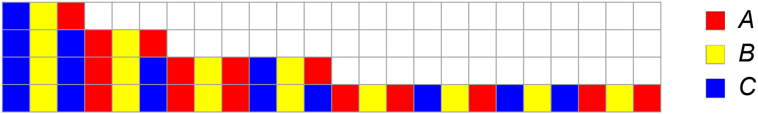
$$\Phi_B : \{A, B, C\} \to \{A, B, C\}; \quad A \mapsto C,\ B \mapsto B,\ C \mapsto A.$$

Set $\mathbf{v}_0 = B$, $\mathbf{v}_1 = CBA$, and for $n \geq 1$ we set $\mathbf{v}_{n+1} := \mathbf{v}_n^C \Phi_B(\mathbf{v}_n^C)$.

Then

$$\mathbf{v}_2 = CBCABA, \quad \mathbf{v}_3 = CBCABC\ ABACBA,$$

$$\mathbf{v}_4 = CBCABC\ ABACBC\ ABACBACBCABA.$$



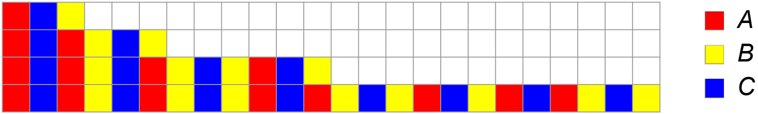
Type-C Thue-Morse words.

$$\Phi_C : \{A, B, C\} \to \{A, B, C\}; \quad A \mapsto B,\ B \mapsto A,\ C \mapsto C.$$

Set $\mathbf{w}_0 = C$, $\mathbf{w}_1 = ACB$, and for $n \geq 1$ we set $\mathbf{w}_{n+1} := \mathbf{w}_n^A \Phi_C(\mathbf{w}_n^A)$. Then

$$\mathbf{w}_2 = ACABCB, \quad \mathbf{w}_3 = ACABCA\,BCBACB,$$

$$\mathbf{w}_4 = ACABCABCBACA\ BCBACBACABCB.$$



Theorem

Let $n \geq 1$. Then

$$\mathbf{U}_{\beta_{n+1}} \setminus \mathbf{U}_{\beta_n} = \{*(\mathbf{u}_n)^\infty, *(\mathbf{v}_n)^\infty, *(\mathbf{w}_n)^\infty\}.$$

Define the generalized Thue-Morse sequence by

$$(\lambda_i) := \lim_{n\to\infty} \phi^n(100) = 101001\,000101\,0001000101001\ldots.$$

Theorem (Main result)

Let $\beta_c \approx 1.55263$ such that $\delta(\beta_c) = (\lambda_i)$. Then β_c is transcendental.

1. *If $\beta \in (\beta_G, \beta_c)$, then U_β is countably infinite;*
2. *If $\beta = \beta_c$, then U_β is uncountable but has zero Hausdorff dimension;*
3. *If $\beta \in (\beta_c, 2)$, then U_β has positive Hausdorff dimension.*

Thank you!