Infinite families of number systems

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Outline

1. GNSs in lattices
2. Infinitely many GNSs with the same radix
3. Sparse alphabets

Number system (GNS)

- A **lattice** ($\mathbb{Z}^d$);
- a **radix** $L \in \mathbb{Z}^{d\times d}$;
- an **alphabet** $\mathcal{A} \subset \mathbb{Z}^d$ containing zero.

Definition

The pair $(L, \mathcal{A})$ is a **number system (GNS)** in $\mathbb{Z}^d$ if every element $x \in \mathbb{Z}^d$ has a unique representation of the form

$$x = \sum_{k=0}^{N} L^k a_k, \qquad a_k \in \mathcal{A}.$$

Note: The binary and the decimal system are not GNS since they do not allow to represent negative numbers. $(-2, \{0,1\})$ is a GNS in $\mathbb{Z}$ and so is $(3, \{-1, 0, 1\})$.

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Necessary conditions

- The radix L must be expansive, i.e. $\rho(L^{-1}) < 1$ (Vince, 1993),
- the alphabet $\mathcal{A}$ must be a complete residue system modulo L,
- $\det(L - I) \neq \pm 1$ (the "unit condition").

Classical question: Given a radix $L \in \mathbb{Z}^{d \times d}$, does there exist an alphabet such that $(L, \mathcal{A})$ is a GNS?

Theorem (Steidl, 1989; Kátai, 1994)

Let $\beta \in \mathbb{Z}[\mathrm{i}]$. Then there exists an alphabet $\mathcal{A}$ such that $(\beta, \mathcal{A})$ is a GNS if and only if $|\beta| \neq 1$, $|\beta - 1| \neq 1$. The same holds in $\mathcal{O}_K$ where K is any imaginary quadratic field.

The used digits lie in a parallelogram around the origin.

Theorem (Germán, Kovács, 2007)

If $\rho(L^{-1}) < 1/2$, then there always exists an alphabet such that $(L, \mathcal{A})$ is a GNS.

They use the **dense** alphabet, i.e. the smallest representative (in a certain vector norm) from every congruence class.
Reminder: $\rho(L^{-1}) < 1$ is a necessary condition.

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Question 1: Given a radix $L \in \mathbb{Z}^{d\times d}$, *how many* alphabets for this radix do exist?

So far known:

- Radix is not expansive or the unit condition fails $\Longrightarrow$ there is no such alphabet.
- For -2 in $\mathbb{Z}$, only the alphabets $\{0,1\}$ and $\{0,-1\}$ are suitable.
- Similarly, for $-1+i$ in $\mathbb{Z}[i]$ there are only four good alphabets.
- Matula, 1982: In $\mathbb{Z}$, for every β with $|\beta| \geq 3$ there are infinitely many alphabets.
- Block diagonal radices $\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$ are easy to handle. (The sublattices can be represented independently – Kovács, 2014 and also Indlekofer, Kátai, Racskó, 1993, etc.)

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Our result:

Theorem

Let L be an operator on a d-dimensional lattice satisfying $\rho(L^{-1}) \leq 1/2$ for which 2 is not an eigenvalue. There always exist infinitely many GNSs with radix L except for the case $d \leq 2$. where $d = 2$ and L has complex eigenvalues (where we do not know), and for the case of radix -2 in $\mathbb{Z}$, where only two GNSs exist.

Note: Except for dimension $d = 2$, the assumptions are weaker than the assumptions of the theorem of Germán and Kovács.

One of the main ingredients:

Proposition

If L is block-triangular, $L = \begin{pmatrix} L_1 & C \\ 0 & L_2 \end{pmatrix}$, and for L_1 and L_2 there exists *one* GNS, then for L there exist *infinitely many* GNSs.

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Sparse alphabets

Question 2: Can all the digits be far away from the origin?

Definition

Given a radix $L \in \mathbb{Z}^{d\times d}$, a sequence of alphabets $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is called a **family of arbitrarily sparse alphabets** if for any given ball B around the origin, there exists an n_B such that for $n \geq n_B$ we have $\mathcal{A}_n \cap B = \{0\}$.

Equivalently we can require that for any finite $0 \notin S \subset \mathbb{Z}^d$, the alphabets $\mathcal{A}_n$ do not use any digits from S for $n \geq n_S$.

If $(L, \mathcal{A}_n)$ is a GNS for every n, we have a **family of arbitrarily sparse GNSs**.

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The final result:

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Suppose that $\rho(L^{-1}) \leq 1/2$ and 2 is not an eigenvalue of L. Then there exists a family of arbitrarily sparse GNSs except for the case when every eigenvalue of L is either an integer or a non-real algebraic number of degree 2, and has geometric multiplicity 1.

Problems arise only if all factors of the characteristic polynomial have degree less than 3.

Again, part of the proof uses the block-triangularisable matrices. But we need the "building stones".

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Theorem

Let $L \in \mathbb{Z}^{d \times d}$ such that

- $\rho(L^{-1}) < 1/2$, *i.e. all eigenvalues of* L *are bigger than* 2;
- L *is diagonalizable over* $\mathbb{C}$;
- *the characteristic polynomial of* L *is* f^k *for some* $k \in \mathbb{N}$ *and* f *is irreducible over* $\mathbb{Q}$;
- *either* $d \geq 3$ *or* L *has real eigenvalues and* $d = 2$.

Then there exists a family of arbitrarily sparse GNSs with radix L.

It is easy to construct a sequence of vector norms $\|\cdot\|_n$ such that the $\|\cdot\|_n$-dense alphabet $\mathcal{A}_n$ gives a GNS for every n. The problem is that quite often all these alphabets are in fact the same.

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Suppose a regular expansive matrix $L \in \mathbb{Z}^{d\times d}$ is diagonalisable over $\mathbb{C}$. Then the following statements are equivalent:

(A) For a basis B let us denote by $\mathcal{A}_n$ the dense alphabet with respect to the vector norm defined as $\|x\|_\infty^{(BD_n)} := \left\|(BD_n)^{-1}x\right\|_\infty$ where $D_n := \mathrm{diag}(1, n, \ldots, n)$. With this notation, the eigenbasis B of L can be chosen so that the sequence of alphabets $\mathcal{A}_n$ forms a family of arbitrarily sparse alphabets with respect to L.

(B) The eigenbasis B can be chosen so that for every congruence class $T_a := L\mathbb{Z}^d + a$ except for T_0 the expression $|[B^{-1}z]_1|$ does not attain a minimum on T_a (it only has an infimum).

(C) There exists an eigenvector b of L^T such that for every congruence class $T_a := L\mathbb{Z}^d + a$ except for T_0 the expression $|b^T z|$ does not attain a minimum on T_a (it only has an infimum).

(D) There exists an eigenvector b of L^T such that if $b^T z = 0$ for $z \in \mathbb{Z}^d$, then $z \equiv 0$ modulo L. Further, the set $\{b^T z : z \in \mathbb{Z}^d\}$ is either dense in $\mathbb{C}$ or a dense subset of $\mathbb{R}$.

(E) There exists an eigenvector b of L^T whose coordinates are linearly independent over $\mathbb{Q}$. Further, either $d \geq 3$ or $d = 2$ and L has only real eigenvalues.

(F) The characteristic polynomial of L is in the form f^k where f is some irreducible integer polynomial. Further, either $d \geq 3$ or $d = 2$ and L has only real eigenvalues.

Starting point:

Theorem (Germán, Kovács, 2007)

If $\rho(L^{-1}) < 1/2$, then a GNS always exists.

Results:

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Suppose that $\rho(L^{-1}) \leq 1/2$ and 2 is not an eigenvalue of L. There always exist infinitely many GNSs with radix L except for the case where $d = 2$ and L has complex eigenvalues (where we do not know), and the case of radix -2 in $\mathbb{Z}$, where only two GNSs exist.

Theorem

Suppose that $\rho(L^{-1}) \leq 1/2$ and 2 is not an eigenvalue of L. Then there exists a family of arbitrarily sparse GNSs except for the case when every eigenvalue of L is either an integer or a non-real algebraic number of degree 2, and has geometric multiplicity 1.

Thank you for your attention!