Ito α-continued fractions and matching

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Joint work (in progress) with Carlo Carminati, Wolfgang Steiner, Hitoshi Nakada

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Universiteit Leiden
The Netherlands

Set up

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Pick $\alpha \in [0,1]$ and let $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ be defined by

$$T_\alpha(x) = \begin{cases} S(x) - \lfloor S(x) + 1 - \alpha \rfloor & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases} \tag{1}$$

Different choices for $S(x)$ in formula (1) give rise to different generalizations of the classical continued fraction algorithms:

(IT) for $S(x) = \frac{1}{x}$ one gets the Ito α-continued fractions first studied by S. Ito and S. Tanaka.

(N) for $S(x) = \frac{1}{|x|}$ one gets the α-continued fractions first studied by H. Nakada.

(KU) for $S(x) = -\frac{1}{x}$ one gets the KU-continued fractions first studied by S. Katok and I. Ugarcovici.

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Briefly how this makes a continued fraction algorithm

For Ito α-continued fractions:

Fix $\alpha \in [0,1]$ and let $d_\alpha(x) = \left\lfloor \frac{1}{x} + 1 - \alpha \right\rfloor$. Then set $d_{\alpha,n}(x) = d_\alpha\left(T_\alpha^n(x)\right)$.This gives

$$T_\alpha(x) = \frac{1}{x} - d_\alpha(x)$$

and so

$$\begin{aligned} x &= \frac{1}{d_\alpha(x) + T_\alpha(x)} \\ &= \frac{1}{d_\alpha(x) + \frac{1}{d_\alpha(T_\alpha(x)) + T_\alpha^2(x)}} \\ &= \frac{1}{d_{\alpha,1} + \frac{1}{d_{\alpha,2} + \ddots}} \end{aligned}$$

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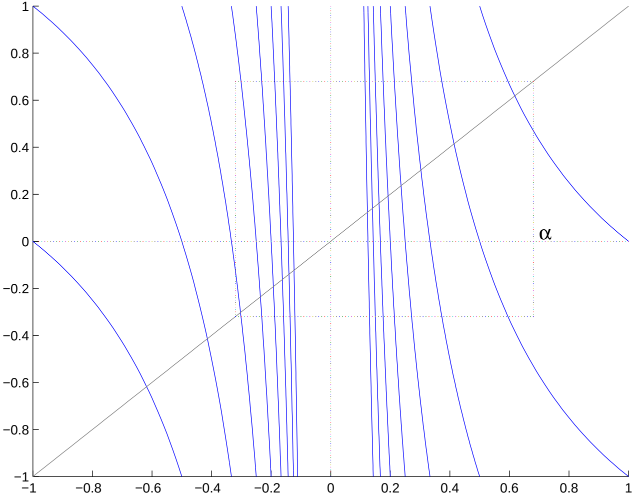
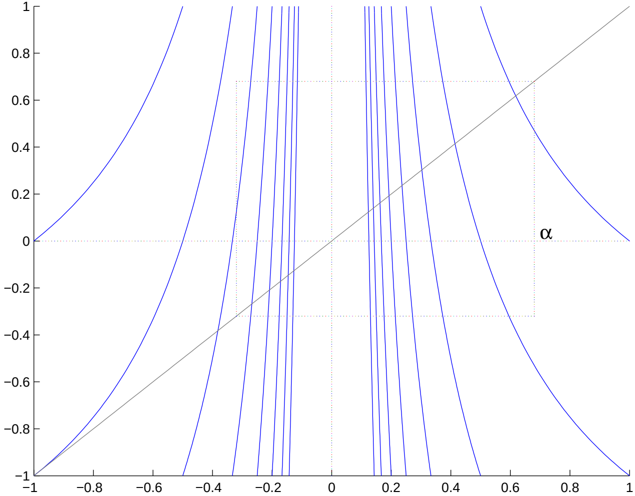


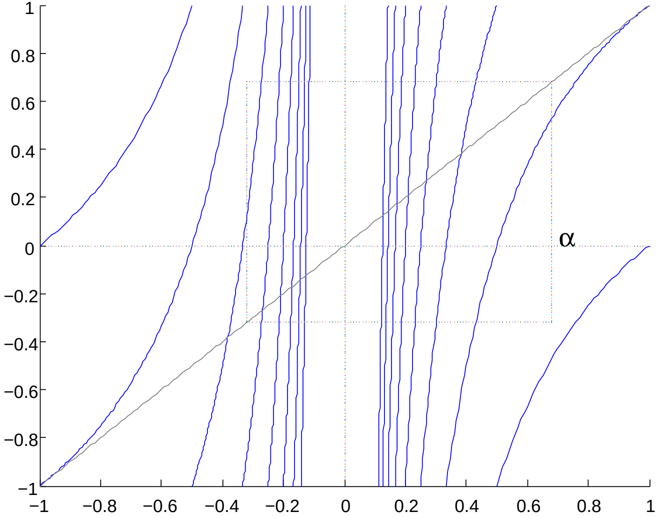
Note: only decreasing branches

Nakada's α-continued fractions



Note: increasing branches for negative x and decreasing branches for positive x

KU-continued fractions



Note: only increasing branches

We are interested in matching

Definition of Matching

A parameter $\alpha \in [0,1]$ satisfies the *matching condition* with *matching exponents* M, N if

$$T_\alpha^N(\alpha) = T_\alpha^M(\alpha - 1). \quad (2)$$

For all three families we have the following properties:

- the matching condition is satisfied for Lebesgue almost every $\alpha \in [0,1]$,
- when taking N, M minimal. For Lebesgue almost every α we can find a neighbourhood of α with the same matching exponents.

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Definitions of matching intervals,matching index, the matching set and the bifurcation set

Let $J \subset [0, 1]$ be an interval with non-empty interior. We say that J is a matching interval with exponents N, M if

(i) condition (2) holds for every $\alpha \in J$;

(ii) N, M are to be taken minimal and are the same for all $x \in J$.

The difference $\Delta := M - N$ is called *matching index*.
We call *matching set* the set A obtained by the union of all matching intervals; its complement will be called *bifurcation set* and will be denoted by $\mathcal{E}$.

Matching intervals and entropy

$$h(\alpha) := h(T_\alpha)$$

Theorem

Let J be a matching interval with matching index Δ. If $\Delta > 0$ then $h(\alpha)$ is increasing on J, if $\Delta < 0$ then $h(\alpha)$ is decreasing on J and when $\Delta = 0$ then $h(\alpha)$ is constant on J.

One can prove this for all three families in the same way by using the fact that for almost every $x \in [\alpha - 1, \alpha]$ we have

$$h(\alpha) = \lim_{n\to\infty} -2\log(|q_{\alpha,n}(x)|)$$

where $q_{\alpha,n}(x)$ is the denominator of the n^{th} convergent of x.

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Some more results on similarities and differences between the families

	Ito	N	KU
matching holds Lebesgue a.e.	Yes	Yes	Yes
on matching intervals $h(\alpha)$ is monotone	Yes	Yes	Yes
$h(\alpha)$ is increasing on $[0, \frac{1}{2}]$	Yes	No	Yes
$\dim_H(\mathcal{E})$	1	1	0
possible matching indices	$\{-2, 0, 2\}$	$\mathbb{Z}$	$\mathbb{Z}$
rational values of α are always contained in a matching interval	No	Yes	Yes

Timeline

- 1981: Ito α-continued fractions introduced by S. Ito and S. Tanaka
- 1981: Nakada's α-continued fractions introduced by H. Nakada
- 2008: H. Nakada and R. Natsui prove for Nakada's α-continued fractions that there are matching intervals on which the entropy is monotone. Results on behavior in the neighborhood of 0. Conjecture matching holds Lebesgue a.e.
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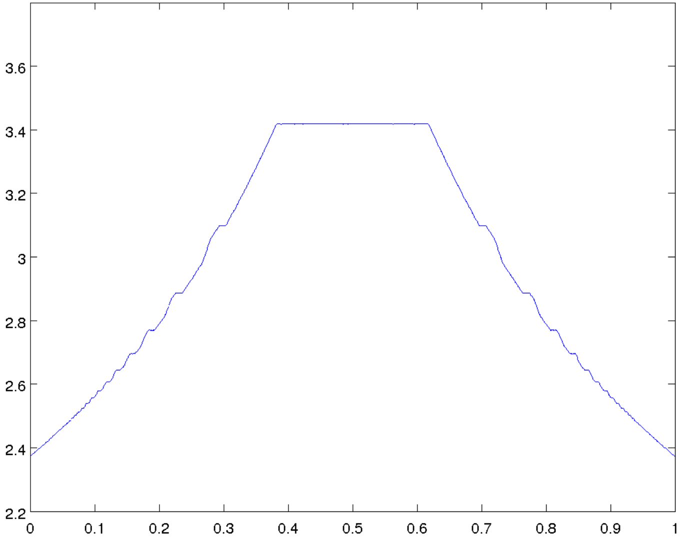
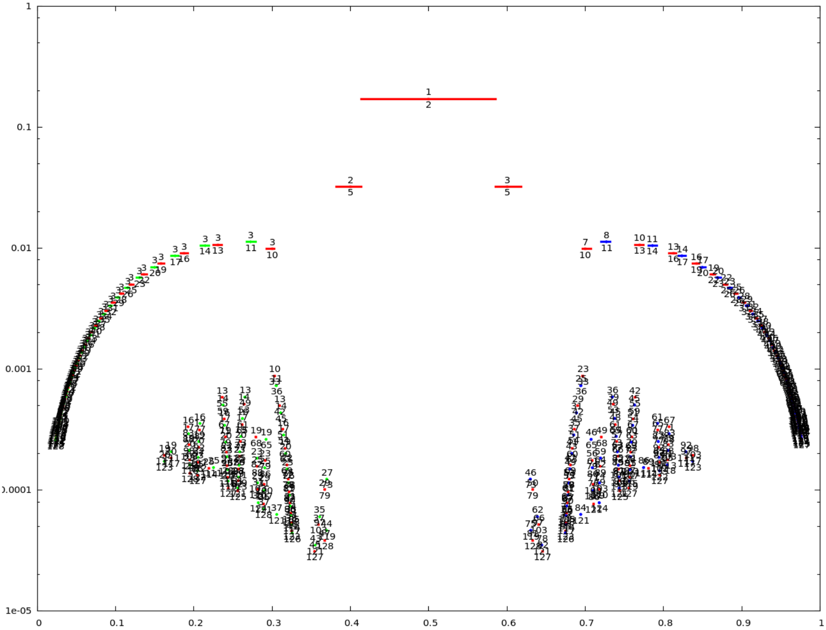
A Simulation

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3
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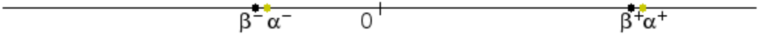
The good and the bad rationals and the ugly?



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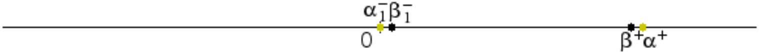

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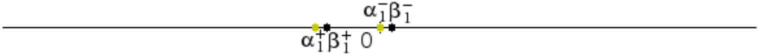

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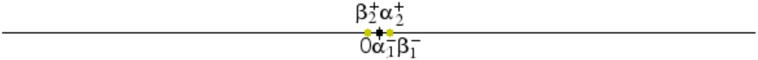

Around a bad rational

Theorem

Let $\frac{1}{2} < r < 1$ *be a rational with odd matching index then the matching index is* 1 *and*

- *there exists an increasing and a decreasing sequence* $(a_n)_{n\geq 1}$ *such that* a_n *has matching index 0 and* $\lim_{n\to\infty} a_n = r$
- *there exists an increasing and a decreasing sequence* $(b_n)_{n\geq 1}$ *such that* b_n *has matching index -2 and* $\lim_{n\to\infty} b_n = r$
- *there exists an increasing and a decreasing sequence* $(c_n)_{n\geq 1}$ *such that* c_n *has matching index 1 and* $\lim_{n\to\infty} c_n = r$

No matching?

Recall that $\mathcal{E} := \{\alpha \in [0,1] :$ there is no matching for $\alpha\}$. We have the following results

- $\dim_h(\mathcal{E}) = 1$
- $\dim_h(\mathcal{E} \cap (g, g+\delta) = 1$ for all $\delta > 0$
- $\dim_h(\mathcal{E} \cap (1-\delta, 1] > \frac{1}{2}$ for all $\delta > 0$

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Sketch of Proof $\dim_h(\mathcal{E}) = 1$

Let $C_n = \{x \in [0,1] : x = [a_1, a_2, \ldots]$ and $a_j, \ldots, a_{j+2n-1} \neq 1^{2n-1}$ for all $j \in \mathbb{N}\}$ and define $\hat{C}_n = 1^{2n+1}2 \circ C_n$.

Let $f_i : [0,1] \to [0,1]$ be defined as $f_i(x) = \frac{1}{i+x}$ with $i \in \mathbb{N}$. Then

- f_i is Bi-libschitz for all $i \in \mathbb{N}$ and so is $f_1^{2n-1} \circ f_2$.
- We have $f_1^{2n+1} \circ f_2(C_n) = \hat{C}_n$ which gives us $\dim_H(C_n) = \dim_H(\hat{C}_n)$

Let $C = \bigcup_{n\in\mathbb{N}}^{\infty} C_n$ and $\hat{C} = \bigcup_{n\in\mathbb{N}}^{\infty} \hat{C}_n$.We find

$$\dim_H(C) = \sup_{n\in\mathbb{N}} \dim_H(C_n) = \sup_{n\in\mathbb{N}} \dim_H(\hat{C}_n) = \dim_H(\hat{C})$$

Let $BAD(g) = \{x \in [0,1] : g \notin \overline{T^n(x) : n \in \mathbb{N}\}}$.
$BAD(g)$ is α-winning and therefor has hausdorff dimension 1.
We have $BAD(g) = C$ and $C \subset \mathcal{E}$.

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$BAD(g)$ is α-winning and therefor has hausdorff dimension 1.
We have $BAD(g) = C$ and $C \subset \mathcal{E}$.

Sketch of Proof $\dim_h(\mathcal{E}) = 1$

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Sketch of Proof $\dim_h(\mathcal{E} \cap (g, g+\delta) = 1$ for all $\delta > 0$

Let $\delta > 0$ then there is an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\hat{C}_n \subset (g, g+\delta)$. This gives us that

$$\bigcup_{n \in \mathbb{N}_{\geq N}} \hat{C}_n \subset (g, g+\delta) \cap \mathcal{E} \tag{3}$$

Since we have that $C_n \subset C_m$ for $n < m$ we find

$$\begin{aligned} \dim_H \bigcup_{n \in \mathbb{N}_{\geq N}} \hat{C}_n &= \sup_{n \in \mathbb{N}_{\geq N}} \dim_H(\hat{C}_n) \\ &= \sup_{n \in \mathbb{N}_{\geq N}} \dim_H(C_n) = \sup_{n \in \mathbb{N}} \dim_H(C_n) = \dim_H(C) = 1. \end{aligned}$$

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For $n > 3$, let $F_n = \{x \in [0,1] : a_i \geq n \text{ for all } i \in \mathbb{N}\}$ and $F_n' = f_1 \circ f_{n-2}(F_n)$. Then $F_n' \subset \mathcal{E}$.

Note that for all $\delta > 0$ there exists an $n > 20$ such that $F_n' \subset (1-\delta, 1]$ and that $\dim_H(F_n) = \dim(F_n')$.

We have that $dim_H(F_n) > \frac{1}{2} + \frac{1}{2\log(n+2)}$ for $n > 20$.

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Thank you