Invariant densities for random systems

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Setting

$(\Omega \subseteq \mathbb{N}, p)$ prob space

$T := \{T_j : X \to X, j \in \Omega\}$ family of maps

T is a **random system** of the space X of probability p, if

$$T(x) := T_j(x) \text{ with probability } p_j$$

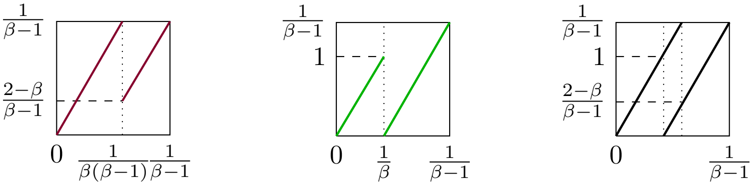
Motivation

- Stochastic perturbations
- Particles systems
- Number expansions
  - β, Lüroth, dyadic expansions, etc.

Motivation

Random β-transformations [DK03]

$$T_0(x) = \begin{cases} \beta x, & \text{if } x \in [0, \frac{1}{\beta(\beta-1)}] \\ \beta x - 1, & \text{if } x \in (\frac{1}{\beta(\beta-1)}, \frac{1}{\beta-1}] \end{cases}, \quad T_1(x) = \begin{cases} \beta x, & \text{if } x \in [0, \frac{1}{\beta}) \\ \beta x - 1, & \text{if } x \in [\frac{1}{\beta}, \frac{1}{\beta-1}] \end{cases}$$



(a) T_0 (b) T_1 (c) T

ACIM: definition

$(X, \mathcal{B}, \mu_p, T, p)$ random system

$$\text{ACIM:} \quad \mu_p(A) = \int_A h \, d\lambda = \sum_{j \in \Omega} p_j \mu_p(T_j^{-1} A) \qquad \text{for all } A \in \mathcal{B}$$

Perron-Frobenius operator

- $\int_A P_T h \, d\lambda = \int_{T^{-1}(A)} h \, d\lambda$
- $P_T h = \sum_{j \in \Omega} P_{T_j} p_j h$

$$P_T h = h \quad \rightsquigarrow \quad \text{ACIM } \mu_h$$

Existing formulas

Lasota-Yorke linear maps

- Same slopes
  - Parry, Dajani, Kempton, Suzuki for the β-transformations (deterministic and random)
- Different slopes
  - Kopf [Kop90], Góra (deterministic)
  - Our approach (random)

Setting

- $T = \{T_j : [0,1] \to [0,1], j \in \Omega\}$ expanding on average wrt p

$$\sup_{x\in[0,1]} \sum_{j\in\Omega} \frac{p_j}{|T_j'(x)|} < 1$$

- T_j piecewise linear
- $\{I_1, ..., I_N\}$ partition for the set $\{0 = z_0 < z_1 < ... < z_N = 1\}$ discontinuity points
- $T_{i,j}(x) = k_{i,j}x + d_{i,j}$
- (Thm, [Ino12]) T admits an ACIM

Assumptions

1. $T(0), T(1) \in \{0, 1\}$

2. for every i there exists n:

$$\frac{\sum_{j\in\Omega} \frac{p_j}{k_{i,j}} d_{i,j}}{1 - \sum_{j\in\Omega} \frac{p_j}{k_{i,j}}} - \frac{\sum_{j\in\Omega} \frac{p_j}{k_{n,j}} d_{n,j}}{1 - \sum_{j\in\Omega} \frac{p_j}{k_{n,j}}} \neq 0$$

Definitions

$\omega \in \Omega^{\mathbb{N}}$ path

$y \in [0,1]$ point

$t \in \mathbb{N}$ instant of time

$n \in \{1, \ldots, N\}$ interval

- $\tau_\omega(y,t) := \frac{p_{\omega_t}}{k_{i,\omega_t}}, \quad$ if $\; T_{\omega_1^{t-1}}(y) \in I_i$
- $\delta_\omega(y,t) := \prod_{n=0}^{t} \tau_\omega(y,n)$
- $\mathrm{KI}_n(y) := \sum_{t \geq 1} \sum_{\omega \in \Omega^t} \delta_\omega(y,t) \mathbf{1}_{I_n}(T_{\omega_1^{t-1}}(y))$

Results

Thm. (Kalle, M., to appear)

Under the previous assumptions,

$$h_\gamma(x) = c \sum_{m=1}^{N-1} \gamma_m \sum_{l\in\Omega} \left[\frac{p_l}{k_{m,l}} L_{a_{m,l}}(x) - \frac{p_l}{k_{m+1,l}} L_{b_{m,l}}(x) \right]$$

is a T-invariant function.

$$a_{m,l} = k_{m,l} z_m + d_{m,l}, \quad b_{m,l} = k_{m+1,l} z_m + d_{m+1,l}$$

$$L_y(x) = \sum_{t\geq 0} \sum_{\omega\in\Omega^t} \delta_\omega(y,t) \mathbf{1}_{[0,T_\omega(y))}(x)$$

Results

Procedure:

$$T \quad \to \quad M \quad \to \quad M\gamma = 0 \quad \to \quad h_\gamma$$

for

$$M = \left(\sum_{j\in\Omega} \left[\frac{p_j}{k_{i,j}} \mathrm{KI}_n(a_{i,j}) - \frac{p_j}{k_{i+1,j}} \mathrm{KI}_n(b_{i,j}) \right] + q_{n,i} \right)_{n,i}$$

Results

Not straightforward

- There always exists $\gamma \neq 0$ (ass. 2.)
- h_γ is T-invariant (ass. 1.)
- $\{I_1, \ldots, I_N\}$ arbitrary (endpoints and size of this set)

Results

Thm. (Kalle, M., to appear)

For Ω finite and T expanding, the construction gives all possible T-invariant densities.

Idea:

$$\begin{array}{ccc} M & \longrightarrow & \hat{M}_U^\dagger \\ \big\downarrow & & \big\downarrow \\ \gamma & \longleftarrow & \hat{\gamma}_U^\dagger \end{array}$$

Example 1: random β-transformations

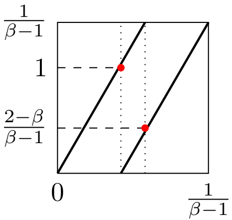
- [DdV07]: special β
- [Kem14]: h for all $1 < \beta < 2$, unbiased case

$$h(x) = c\sum_{n=0}^{\infty}\frac{1}{(2\beta)^n}\left(\sum_{\omega_1\cdots\omega_n\in\{0,1\}^n}\mathbf{1}_{[0,R^n_{\beta,\omega_1\cdots\omega_n}(1)]}(x)+\mathbf{1}_{[R^n_{\beta,\omega_1\cdots\omega_n}(\frac{2-\beta}{\beta-1}),\frac{1}{\beta-1}]}(x)\right)$$

- [Suz17]: h for all $1 < \beta < 2$, biased cases

Example 1: random β-transformations

- h for all $1 < \beta < 2$ for $p_0 = p_1 = \frac{1}{2}$



$$h_\gamma(x) = k \sum_{t\geq 0} \sum_{\omega\in\{0,1\}^t} \frac{1}{(2\beta)^t} \left(\mathbf{1}_{[0,T_\omega(1))}(x) + \mathbf{1}_{[T_\omega(\frac{2-\beta}{\beta-1}),\frac{1}{\beta-1}]}(x) \right)$$

Example 1: random β-transformations

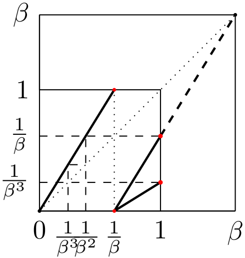
$\mathrm{KI}_n(1)$	$\mathrm{KI}_n\left(\frac{2-\beta}{\beta-1}\right)$	$\mathrm{KI}_n(0)$	$\mathrm{KI}_n\left(\frac{1}{\beta-1}\right)$
c_1	c_3	$\frac{1}{\beta-1}$	0
c_2	c_2	0	0
c_3	c_1	0	$\frac{1}{\beta-1}$

$$\downarrow$$

$$\begin{pmatrix} \frac{1}{\beta} + \frac{1}{2\beta}(c_1 - \frac{1}{\beta-1}) & -\frac{1}{2\beta}c_3 \\ -\frac{1}{\beta} + \frac{1}{2\beta}c_2 & \frac{1}{\beta} - \frac{1}{2\beta}c_2 \\ \frac{1}{2\beta}c_3 & -\frac{1}{\beta} - \frac{1}{2\beta}(c_1 - \frac{1}{\beta-1}) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

Example 2: random (α, β)-transformations

$$T_0(x) = \begin{cases} \beta x, & \text{if } x \in [0, 1/\beta) \\ \frac{\alpha}{\beta}(\beta x - 1), & \text{if } x \in [1/\beta, 1] \end{cases} \quad \text{and} \quad T_1(x) = \begin{cases} \beta x, & \text{if } x \in [0, 1/\beta) \\ \beta x - 1, & \text{if } x \in [1/\beta, 1] \end{cases}$$

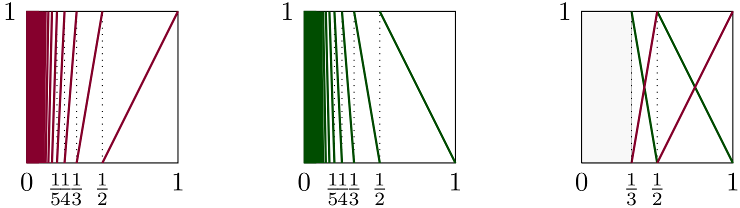


$$\beta^2 = \beta + 1,\ \alpha = 1/\beta \quad \rightarrow \quad h_\gamma = c((\beta^2 p + \beta)\mathbf{1}_A + (p + \beta)\mathbf{1}_B + \beta\mathbf{1}_C + 1\mathbf{1}_D)$$

[DHK09]

Example 3: random Lüroth map with bounded digits

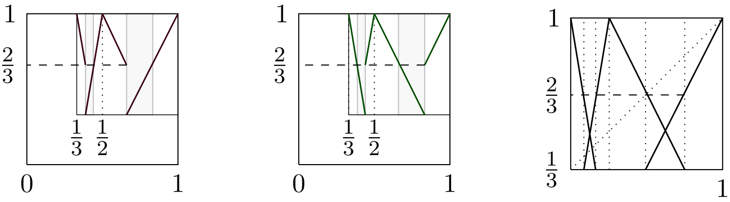
$$x = \sum_{n \geq 1} (-1)^{s_{n-1}} (r_n + \omega_n - 1) \prod_{k=1}^{n} \frac{1}{r_k(r_k - 1)}$$



$$T_L(x) := n(n-1)x - (n-1) \quad \text{and} \quad T_A(x) := 1 - T_L(x)$$

[Lür83, BBDK94, BI09, LY78, Pel84]

Example 3: random Lüroth map with bounded digits



(g) T_0 (h) T_1 (i) T

$$h_\gamma(x) = 3/8(3 \cdot \mathbf{1}_{[\frac{1}{3},\frac{2}{3}]}(x) + 5 \cdot \mathbf{1}_{(\frac{2}{3},1]}(x))$$

- digits frequency: $2 \to 13/16$, $3 \to 3/16$

Thank you!

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