Some complexity results in the theory of normal numbers

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Examples:

$\mathbb{R}, 2^\mathbb{N}, b^\mathbb{N}, 2^{\mathbb{N} \times \mathbb{N}}$
In any topological space $X$, the collection of Borel sets $\mathcal{B}(X)$ is the smallest $\sigma$-algebra containing the open sets. They are stratified into levels, the Borel hierarchy, by defining $\Sigma^0_1$ = the open sets, $\Pi^0_1 = \neg \Sigma^0_1 = \{X - A: A \in \Sigma^0_1\}$ = the closed sets, and for $\alpha < \omega_1$ we let $\Sigma^0_\alpha$ be the collection of countable unions $A = \bigcup_n A_n$ where each $A_n \in \Pi^0_{\alpha_n}$ for some $\alpha_n < \alpha$. We also let $\Pi^0_\alpha = \neg \Sigma^0_\alpha$.

Alternatively, $A \in \Pi^0_\alpha$ if $A = \bigcap_n A_n$ where $A_n \in \Sigma^0_{\alpha_n}$ where each $\alpha_n < \alpha$. We also set $\Delta^0_\alpha = \Pi^0_\alpha \cap \Sigma^0_\alpha$, in particular $\Delta^0_1$ is the collection of clopen sets. For any topological space, $\mathcal{B}(X) = \bigcup_{\alpha < \omega_1} \Sigma^0_\alpha = \bigcup_{\alpha < \omega_1} \Pi^0_\alpha$. All of the collections $\Delta^0_\alpha$, $\Sigma^0_\alpha$, $\Pi^0_\alpha$ are pointclasses, that is, they are closed under inverse images of continuous functions.
The Borel Hierarchy

In any topological space $X$, the collection of Borel sets $\mathcal{B}(X)$ is the smallest $\sigma$-algebra containing the open sets. They are stratified into levels, the Borel hierarchy, by defining $\Sigma^0_1 = \text{the open sets}$, $\Pi^0_1 = \neg \Sigma^0_1 = \{X - A : A \in \Sigma^0_1\} = \text{the closed sets}$, and for $\alpha < \omega_1$ we let $\Sigma^0_\alpha$ be the collection of countable unions $A = \bigcup_n A_n$ where each $A_n \in \Pi^0_{\alpha_n}$ for some $\alpha_n < \alpha$. We also let $\Pi^0_\alpha = \neg \Sigma^0_\alpha$.

Alternatively, $A \in \Pi^0_\alpha$ if $A = \bigcap_n A_n$ where $A_n \in \Sigma^0_{\alpha_n}$ where each $\alpha_n < \alpha$. We also set $\Delta^0_\alpha = \Pi^0_\alpha \cap \Sigma^0_\alpha$, in particular $\Delta^0_1$ is the collection of clopen sets. For any topological space, $\mathcal{B}(X) = \bigcup_{\alpha < \omega_1} \Sigma^0_\alpha = \bigcup_{\alpha < \omega_1} \Pi^0_\alpha$. All of the collections $\Delta^0_\alpha$, $\Sigma^0_\alpha$, $\Pi^0_\alpha$ are pointclasses, that is, they are closed under inverse images of continuous functions.

For example, $\Sigma^0_2$ consists of $F_\sigma$ sets and $\Pi^0_2$ consists of $G_\delta$ sets. $\Pi^0_3$ contains the sets which are intersections of $F_\sigma$ sets.
A fundamental result of Suslin says that in any Polish space $\mathcal{B}(X) = \Delta_1^1 = \Sigma_1^1 \cap \Pi_1^1$, where $\Pi_1^1 = \neg \Sigma_1^1$, and $\Sigma_1^1$ is the pointclass of continuous images of Borel sets. Equivalently, $A \in \Sigma_1^1$ iff $A$ can be written as $x \in a \leftrightarrow \exists y \ (x, y) \in B$ where $B \subseteq X \times Y$ is Borel (for some Polish space $Y$). Similarly, $A \in \Pi_1^1$ iff it is of the form $x \in A \leftrightarrow \forall y \ (x, y) \in B$ for a Borel $B$. The $\Sigma_1^1$ sets are also called the analytic sets, and $\Pi_1^1$ the co-analytic sets. We also have $\Sigma_1^1 \neq \Pi_1^1$ for any uncountable Polish space.
The Borel Hierarchy

\[ \Delta^0_1 = \text{clopen} \]
\[ \Pi^1_0 = \text{closed} \]
\[ \Sigma^0_0 = \text{open} \]
\[ \Sigma^0_2 = F_\sigma \]
\[ \Sigma^0_3 \]
\[ \delta \]
\[ \Delta^0_2 \]
\[ \Delta^0_3 \]
\[ \cdots \]
\[ \Delta^1_1 \]
\[ \Pi^1_3 \]
\[ \Pi^1_1 = \text{coanalytic} \]
\[ \Sigma^1_1 = \text{analytic} \]
A basic fact is that for any uncountable Polish space $X$, there is no collapse in the levels of the Borel hierarchy, that is, all the various pointclasses $\Delta^0_\alpha$, $\Sigma^0_\alpha$, $\Pi^0_\alpha$, for $\alpha < \omega_1$, are all distinct. Thus, these levels of the Borel hierarchy can be used to calibrate the descriptive complexity of a set. We say a set $A \subseteq X$ is $\Sigma^0_\alpha$ (resp. $\Pi^0_\alpha$) hard if $A \notin \Pi^0_\alpha$ (resp. $A \notin \Sigma^0_\alpha$). This says $A$ is “no simpler” than a $\Sigma^0_\alpha$ set. We say $A$ is $\Sigma^0_\alpha$-complete if $A \in \Sigma^0_\alpha - \Pi^0_\alpha$, that is, $A \in \Sigma^0_\alpha$ and $A$ is $\Sigma^0_\alpha$ hard. This says $A$ is exactly at the complexity level $\Sigma^0_\alpha$. Likewise, $A$ is $\Pi^0_\alpha$-complete if $A \in \Pi^0_\alpha - \Sigma^0_\alpha$. 
Examples of Borel sets

In $\mathbb{R}$, $(a, b) \in \Sigma^1_0$, $[a, b] \in \Pi^1_0$, and $\mathbb{R} \in \Delta^1_0$. 

Note that $Q$ is $\Sigma^0_2$-complete, $R \setminus Q$ is $\Pi^0_2$-complete, but $\emptyset \in \Delta^1_0$.

The set of normal numbers is $\Pi^0_3$-complete (Ki and Linton 1994) and the set of absolutely normal numbers is also $\Pi^0_3$-complete (Becher, Heiber, and Slaman 1994).

The set of essentially nonnormal numbers is $\Sigma^0_3$-complete (Airey, Kwietniak, Jackson, and M. in preparation).

Roman Nikiforov will explain essentially nonnormal numbers in the next talk.

Let $X = C([0, 1])$ with the sup norm. If $S = \{f \in X : f$ is nowhere differentiable $\}$, then $S \in \Pi^1_1 \setminus \Sigma^1_1$ (R. D. Mauldin 1979).
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$\emptyset \in \Sigma_0^2$, $\mathbb{R} \setminus \emptyset \in \Pi_2^0$, and $\emptyset = \emptyset \cap \mathbb{R} \setminus \emptyset \in \Delta_2^0$. 

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Let $\mathcal{N}(b)$ be the set of numbers normal in base $b$ and let

$$\mathcal{N}^\perp(b) = \{y : \forall x \in \mathcal{N}(b) \ (x + y) \in \mathcal{N}(b)\}.$$

be the set of numbers that preserve normality in base $b$ under addition.
Preserving normality under addition

Let $\mathcal{N}(b)$ be the set of numbers normal in base $b$ and let

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It can easily be shown that $\mathbb{Q} \subseteq \mathcal{N}^\perp(b)$. But it is not clear if $\mathcal{N}^\perp(b)$ is even a Borel set.
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G. Rauzy gave a complete characterization of $\mathcal{N}^\perp(b)$ in 1976. As a consequence of this characterization, it can be shown that $\mathcal{N}^\perp(b)$ is Borel and that $\mathcal{N}^\perp(b) \in \Pi^0_3$. 

Some complexity results
For any positive integer length $\ell$, let $\mathcal{E}_\ell$ denote the set of functions from $b^\ell$ to $b$. We call an $E \in \mathcal{E}_\ell$ a block function of width $\ell$. 
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We set, for \( x \in \mathbb{R} \),

\[
\beta_\ell(x, N) = \inf_{E \in \mathcal{E}_\ell} \frac{1}{N} \sum_{n < N} \inf \{ 1, |c_n - E(c_{n+1}, \ldots, c_{n+\ell})| \},
\]

where \( x = 0.c_0c_1c_2\ldots \) in base \( b \).
Preserving normality under addition

We then define the lower and upper noises $\beta^-(x)$, $\beta^+(x)$ of $x$ by:

$$\beta^-(x) = \lim_{\ell \to \infty} \beta^-_{\ell}(x),$$

where

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The upper noise $\beta^+(x)$ is defined similarly using

$$\beta^+(x) = \lim_{\ell \to \infty} \beta^+_{\ell}(x)$$

where

$$\beta^+_{\ell}(x) = \limsup_{N \to \infty} \beta_{\ell}(x, N).$$
Let $s \in \left[0, \frac{b-1}{b}\right]$. Let

\[
A_1(s) = \{x : \beta^-(x) \leq s\}, \quad A_2(s) = \{x : \beta^-(x) \geq s\}
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\[
A_3(s) = \{x : \beta^+(x) \leq s\}, \quad A_4(s) = \{x : \beta^+(x) \geq s\}
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Theorem (G. Rauzy 1976) $\mathcal{N}^\perp(b) = A_3(0)$ and $\mathcal{N}(b) = A_2\left(\frac{b-1}{b}\right)$. 

Note that $\beta^+ + (x) = 0$ is equivalent to $\forall \varepsilon \in \mathbb{Q}^+ \exists M, L \in \mathbb{N}$ such that $\forall N > M, \ell > L$ we have $\beta^-(x, N) < \varepsilon$. Thus, it follows by Rauzy's theorem that $\mathcal{N}^\perp(b)$ is a Borel set and is a $\Pi_{30}$ set.
Preserving normality under addition

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Thus, it follows by Rauzy’s theorem that $\mathcal{N}^\perp(b)$ is a Borel set and is a $\Pi^3_0$ set.
**Theorem** (Airey, Jackson, M.) For any $s \in [0, \frac{b-1}{b})$, the set $A_1(s)$ is \( \Pi_4^0 \)-complete and the set $A_3(s)$ is \( \Pi_3^0 \)-complete. For any $s \in (0, \frac{b-1}{b}]$, the set $A_2(s)$ is \( \Pi_3^0 \)-complete, and the set $A_4(s)$ is \( \Pi_2^0 \)-complete.
Complexity

**Theorem** (Airey, Jackson, M.) For any $s \in [0, \frac{b-1}{b}]$, the set $A_1(s)$ is $\Pi^0_4$-complete and the set $A_3(s)$ is $\Pi^0_3$-complete. For any $s \in (0, \frac{b-1}{b}]$, the set $A_2(s)$ is $\Pi^0_3$-complete, and the set $A_4(s)$ is $\Pi^0_2$-complete.

Put $H(s) = -s \log s - (1-s) \log(1-s)$. For $s \in [0, \frac{b-1}{b}]$ we have

$$
\dim_H(A_1(s)) = 1 \\
\dim_H(A_2(s)) = 1 \\
\frac{1}{\log b} H(s) + \frac{\log(b-1)}{\log b} s \leq \dim_H(A_3(s)) \leq \frac{1}{\log b} H(s) + s \\
\dim_H(A_4(s)) = 1.
$$
Let $X$ and $Y$ be Polish spaces and let $A \subseteq X$ and $B \subseteq Y$ along with a continuous function $f : Y \to X$ where $f^{-1}(A) = B$. Then if $B$ is $\Sigma_\alpha^0$-complete (resp. $\Pi_\alpha^0$-complete), then $A$ is $\Sigma_\alpha^0$-hard ($\Pi_\alpha^0$-hard).
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The function $f$ reduces the question of membership in $A$ to membership in $B$. 

**Some complexity results**
Wadge reduction

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We will use Wadge reduction to prove that for all $s \in [0, \frac{b-1}{b})$, the set $A_3(s)$ is $\Pi^0_3$-hard and thus $\Pi^0_3$-complete.
The proof that $A_3(s)$ is $\Pi^0_3$-hard

The set

$$P = \left\{ x = (x_{ij}) \in 2^{\mathbb{N} \times \mathbb{N}} : \forall i \exists j_0 \ \forall j \geq j_0 \ x_{ij} = 0 \right\}$$

is $\Pi^0_3$-complete. Thus, we will construct a mapping from $Y = 2^{\mathbb{N} \times \mathbb{N}}$ to $X = \mathbb{R}$ where each member of $P$ is mapped to a number with upper noise $\leq s$ and each member not in $P$ is mapped to a number with upper noise $> s$. 

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Some complexity results
The proof that $A_3(s)$ is $\Pi^0_3$-hard

Let

$$\mathbf{p}_n = (p_{n,0}, p_{n,1}, \ldots, p_{n,b-1})$$

be a probability vector such that

$$1 - \max_{0 \leq d \leq b-1} p_{n,d} = 1 - p_{n,0} = s + \frac{1 - s - 1/b}{n},$$
The proof that $A_3(s)$ is $\Pi^0_3$-hard

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$$1 - \max_{0 \leq d \leq b-1} p_{n,d} = 1 - p_{n,0} = s + \frac{1 - s - 1/b}{n},$$

so that 0 is the most common digit in the Besicovitch-Eggleston set corresponding to $p_n$. Let $u_n$ be any real number in this set with independent digits.
The proof that $A_3(s)$ is $\Pi^0_3$-hard

It can be shown that $\beta(u_n) = s + \frac{1-s-1/b}{n}$ by using the constant zero function in the definition of noise.
The proof that $A_3(s)$ is $\Pi^0_3$-hard

It can be shown that $\beta(u_n) = s + \frac{1-s-1/b}{n}$ by using the constant zero function in the definition of noise.

Key properties

1. $\lim_{n \to \infty} \beta(u_n) = s$
2. $\beta(u_n) > s \ \forall n$

Next, partition $\mathbb{N}$ into consecutive blocks $B_1, B_2, \ldots$ whose length is growing VERY rapidly.
The proof that \( A_3(s) \) is \( \Pi^0_3 \)-hard

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Let \( \sigma \in 2^{\mathbb{N} \times \mathbb{N}} \) and \( m(\sigma, n) = \min\{ i : \sigma_{i,n} = 1 \} \).
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2. $\beta(u_n) > s \; \forall n$

Next, partition $\mathbb{N}$ into consecutive blocks $B_1, B_2, \ldots$ whose length is growing VERY rapidly.

Let $\sigma \in 2^{\mathbb{N} \times \mathbb{N}}$ and $m(\sigma, n) = \min\{i : \sigma_{i,n} = 1\}$.

So $\sigma \in P$ if and only if $\lim \inf_{n \to \infty} m(\sigma, n) = \infty$. 
The proof that $A_3(s)$ is $\Pi^0_3$-hard

We now construct the continuous mapping $\pi : 2^{\mathbb{N} \times \mathbb{N}} \to \mathbb{R}$ as follows

$$\pi(\sigma)(k) = u_{m(\sigma,n)} \left( k - \sum_{i=1}^{n-1} \#B_i \right).$$
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Thus, we are writing down digits from the \( m(\sigma, n) \)'th member of the sequence \( (u_i) \) in \( B_n \).
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So,

$$\beta^+(\pi(\sigma)) = \limsup_{n \to \infty} \beta^+ (u_{m(\sigma, n)}).$$
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So,

$$\beta^+(\pi(\sigma)) = \limsup_{n \to \infty} \beta^+(u_{m(\sigma,n)})$$

If $\sigma \in P$, $m(\sigma,n) \to \infty$, so $\beta^+(\pi(\sigma)) = s$.

If $\sigma \notin P$, $m(\sigma,n) \not\to \infty$, so $\beta^+(\pi(\sigma)) > s$.
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We now construct the continuous mapping $\pi : 2^\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ as follows

$$
\pi(\sigma)(k) = u_{m(\sigma,n)} \left( k - \sum_{i=1}^{n-1} \#B_i \right).
$$

Thus, we are writing down digits from the $m(\sigma,n)$’th member of the sequence $(u_i)$ in $B_n$.

So,

$$
\beta^+(\pi(\sigma)) = \limsup_{n \rightarrow \infty} \beta^+(u_{m(\sigma,n)})
$$

If $\sigma \in P$, $m(\sigma,n) \rightarrow \infty$, so $\beta^+(\pi(\sigma)) = s$.

If $\sigma \notin P$, $m(\sigma,n) \not\rightarrow \infty$, so $\beta^+(\pi(\sigma)) > s$.

The function $\pi$ depends on only finitely many digits so it is continuous. Thus, $A_3(s)$ is $\Pi^0_3$-hard.