Some complexity results in the theory of normal numbers

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Examples:

$$\mathbb{R}, 2^{\mathbb{N}}, b^{\mathbb{N}}, 2^{\mathbb{N}\times\mathbb{N}}$$

The Borel Hierarchy

In any topological space X, the collection of Borel sets $\mathcal{B}(X)$ is the smallest σ-algebra containing the open sets. They are stratified into levels, the Borel hierarchy, by defining $\boldsymbol{\Sigma}_1^0 =$ the open sets, $\boldsymbol{\Pi}_1^0 = \neg\boldsymbol{\Sigma}_1^0 = \{X - A \colon A \in \boldsymbol{\Sigma}_1^0\} =$ the closed sets, and for $\alpha < \omega_1$ we let $\boldsymbol{\Sigma}_\alpha^0$ be the collection of countable unions $A = \bigcup_n A_n$ where each $A_n \in \boldsymbol{\Pi}_{\alpha_n}^0$ for some $\alpha_n < \alpha$. We also let $\boldsymbol{\Pi}_\alpha^0 = \neg\boldsymbol{\Sigma}_\alpha^0$. Alternatively, $A \in \boldsymbol{\Pi}_\alpha^0$ if $A = \bigcap_n A_n$ where $A_n \in \boldsymbol{\Sigma}_{\alpha_n}^0$ where each $\alpha_n < \alpha$. We also set $\boldsymbol{\Delta}_\alpha^0 = \boldsymbol{\Pi}_\alpha^0 \cap \boldsymbol{\Sigma}_\alpha^0$, in particular $\boldsymbol{\Delta}_1^0$ is the collection of clopen sets. For any topological space, $\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \boldsymbol{\Sigma}_\alpha^0 = \bigcup_{\alpha<\omega_1} \boldsymbol{\Pi}_\alpha^0$. All of the collections $\boldsymbol{\Delta}_\alpha^0$, $\boldsymbol{\Sigma}_\alpha^0$, $\boldsymbol{\Pi}_\alpha^0$ are pointclasses, that is, they are closed under inverse images of continuous functions.

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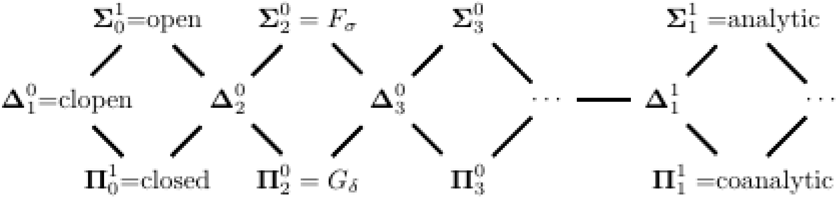
For example, $\mathbf{\Sigma}^0_2$ consists of F_σ sets and $\mathbf{\Pi}^0_2$ consists of G_δ sets. $\mathbf{\Pi}^0_3$ contains the sets which are intersections of F_σ sets.

The Borel Hierarchy

A fundamental result of Suslin says that in any Polish space $\mathcal{B}(X) = \mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$, where $\mathbf{\Pi}^1_1 = \neg\mathbf{\Sigma}^1_1$, and $\mathbf{\Sigma}^1_1$ is the pointclass of continuous images of Borel sets. Equivalently, $A \in \mathbf{\Sigma}^1_1$ iff A can be written as $x \in a \leftrightarrow \exists y\ (x, y) \in B$ where $B \subseteq X \times Y$ is Borel (for some Polish space Y). Similarly, $A \in \mathbf{\Pi}^1_1$ iff it is of the form $x \in A \leftrightarrow \forall y\ (x, y) \in B$ for a Borel B. The $\mathbf{\Sigma}^1_1$ sets are also called the *analytic* sets, and $\mathbf{\Pi}^1_1$ the *co-analytic sets*. We also have $\mathbf{\Sigma}^1_1 \neq \mathbf{\Pi}^1_1$ for any uncountable Polish space.

The Borel Hierarchy



The Borel Hierarchy

A basic fact is that for any uncountable Polish space X, there is no collapse in the levels of the Borel hierarchy, that is, all the various pointclasses $\mathbf{\Delta}^0_\alpha$, $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, for $\alpha < \omega_1$, are all distinct. Thus, these levels of the Borel hierarch can be used to calibrate the descriptive complexity of a set. We say a set $A \subseteq X$ is $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) *hard* if $A \notin \mathbf{\Pi}^0_\alpha$ (resp. $A \notin \mathbf{\Sigma}^0_\alpha$). This says A is "no simpler" than a $\mathbf{\Sigma}^0_\alpha$ set. We say A is $\mathbf{\Sigma}^0_\alpha$-*complete* if $A \in \mathbf{\Sigma}^0_\alpha - \mathbf{\Pi}^0_\alpha$, that is, $A \in \mathbf{\Sigma}^0_\alpha$ and A is $\mathbf{\Sigma}^0_\alpha$ hard. This says A is exactly at the complexity level $\mathbf{\Sigma}^0_\alpha$. Likewise, A is $\mathbf{\Pi}^0_\alpha$-complete if $A \in \mathbf{\Pi}^0_\alpha - \mathbf{\Sigma}^0_\alpha$.

Examples of Borel sets

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Let $X = C([0, 1])$ with the sup norm. If $S = \{f \in X : f \text{ is nowhere differentiable}\}$, then $S \in \mathbf{\Pi}^1_1\backslash\mathbf{\Sigma}^1_1$ (R. D. Mauldin 1979).

Preserving normality under addition

Let $\mathcal{N}(b)$ be the set of numbers normal in base b and let

$$\mathcal{N}^{\perp}(b) = \{y \colon \forall x \in \mathcal{N}(b)\ (x + y) \in \mathcal{N}(b)\}.$$

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G. Rauzy gave a complete characterization of $\mathcal{N}^{\perp}(b)$ in 1976. As a consequence of this characterization, it can be shown that $\mathcal{N}^{\perp}(b)$ is Borel and that $\mathcal{N}^{\perp}(b) \in \mathbf{\Pi}^0_3$.

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For any positive integer length ℓ, let $\mathcal{E}_\ell$ denote the set of functions from b^ℓ to b. We call an $E \in \mathcal{E}_\ell$ a *block function* of width ℓ.

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We set, for $x \in \mathbb{R}$,

$$\beta_\ell(x, N) = \inf_{E \in \mathcal{E}_\ell} \frac{1}{N} \sum_{n<N} \inf\{1, |c_n - E(c_{n+1}, \ldots, c_{n+\ell})|\},$$

where $x = 0.c_0 c_1 c_2 \ldots$ in base b.

Preserving normality under addition

We then define the lower and upper noises $\beta^-(x)$, $\beta^+(x)$ of x by:

$$\beta^-(x) = \lim_{\ell \to \infty} \beta_\ell^-(x),$$

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The upper noise $\beta^+(x)$ is defined similarly using

$$\beta^+(x) = \lim_{\ell \to \infty} \beta_\ell^+(x)$$

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Preserving normality under addition

Let $s \in \left[0, \frac{b-1}{b}\right]$. Let

$$A_1(s) = \{x \colon \beta^-(x) \le s\}, \quad A_2(s) = \{x \colon \beta^-(x) \ge s\}$$
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Thus, it follows by Rauzy's theorem that $\mathcal{N}^\perp(b)$ is a Borel set and is a $\mathbf{\Pi}_0^3$ set.

Complexity

Theorem(Airey, Jackson, M.) For any $s \in \left[0, \frac{b-1}{b}\right)$, the set $A_1(s)$ is $\mathbf{\Pi}^0_4$-complete and the set $A_3(s)$ is $\mathbf{\Pi}^0_3$-complete. For any $s \in (0, \frac{b-1}{b}]$, the set $A_2(s)$ is $\mathbf{\Pi}^0_3$-complete, and the set $A_4(s)$ is $\mathbf{\Pi}^0_2$-complete.

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Put $H(s) = -s\log s - (1-s)\log(1-s)$. For $s \in \left[0, \frac{b-1}{b}\right]$ we have

$$\begin{aligned}
\dim_H(A_1(s)) &= 1\\
\dim_H(A_2(s)) &= 1\\
\frac{1}{\log b}H(s) + \frac{\log(b-1)}{\log b}s \leq \dim_H(A_3(s)) &\leq \frac{1}{\log b}H(s) + s\\
\dim_H(A_4(s)) &= 1.
\end{aligned}$$

Wadge reduction

Let X and Y be Polish spaces and let $A \subseteq X$ and $B \subseteq Y$ along with a continuous function $f : Y \to X$ where $f^{-1}(A) = B$. Then if B is $\mathbf{\Sigma}^0_\alpha$-complete (resp. $\mathbf{\Pi}^0_\alpha$-complete), then A is $\mathbf{\Sigma}^0_\alpha$-hard ($\mathbf{\Pi}^0_\alpha$-hard).

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We will use Wadge reduction to prove that for all $s \in \left[0, \frac{b-1}{b}\right)$, the set $A_3(s)$ is $\mathbf{\Pi}^0_3$-hard and thus $\mathbf{\Pi}^0_3$-complete.

The proof that $A_3(s)$ is $\mathbf{\Pi}^0_3$-hard

The set

$$P = \left\{ x = (x_{ij}) \in 2^{\mathbb{N}\times\mathbb{N}} : \forall i \ \exists j_0 \ \forall j \geq j_0 \ x_{ij} = 0 \right\}$$

is $\mathbf{\Pi}^0_3$-complete. Thus, we will construct a mapping from $Y = 2^{\mathbb{N}\times\mathbb{N}}$ to $X = \mathbb{R}$ where each member of P is mapped to a number with upper noise $\leq s$ and each member not in P is mapped to a number with upper noise $> s$.

The proof that $A_3(s)$ is $\mathbf{\Pi}_3^0$-hard

Let

$$\mathbf{p}_n = (p_{n,0}, p_{n,1}, \ldots, p_{n,b-1})$$

be a probability vector such that

$$1 - \max_{0 \le d \le b-1} p_{n,d} = 1 - p_{n,0} = s + \frac{1 - s - 1/b}{n},$$

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so that 0 is the most common digit in the Besicovitch-Eggleston set corresponding to $\mathbf{p}_n$. Let u_n be any real number in this set with independent digits.

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It can be shown that $\beta(u_n) = s + \frac{1-s-1/b}{n}$ by using the constant zero function in the definition of noise.

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Key properties

1. $\lim_{n\to\infty} \beta(u_n) = s$
2. $\beta(u_n) > s \ \forall n$

Next, partition $\mathbb{N}$ into consecutive blocks $B_1, B_2, \ldots$ whose length is growing VERY rapidly.

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So $\sigma \in P$ if and only if $\liminf_{n\to\infty} m(\sigma, n) = \infty$.

The proof that $A_3(s)$ is $\mathbf{\Pi}_3^0$-hard

We now construct the continuous mapping $\pi : 2^{\mathbb{N}\times\mathbb{N}} \to \mathbb{R}$ as follows

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If $\sigma \in P$, $m(\sigma, n) \to \infty$, so $\beta^+(\pi(\sigma)) = s$.
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The function π depends on only finitely many digits so it is continuous. Thus, $A_3(s)$ is $\mathbf{\Pi}_3^0$-hard.