

Möbius orthogonality for automatic sequences and beyond

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May 24, 2018

Joint work with Michael Drmota and Lukas Spiegelhofer

Möbius function

The Möbius function is defined by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is squarefree and} \\ & k \text{ is the number of prime factors} \\ 0 & \text{otherwise} \end{cases}$$

A sequence \mathbf{u} is **orthogonal to the Möbius function** $\mu(n)$ if

$$\sum_{n \leq N} \mu(n) u_n = o\left(\sum_{n \leq N} |u_n|\right) \quad (N \rightarrow \infty).$$

Old Heuristic - Mobius Randomness Law

Any "reasonably defined (easy)" bounded sequence independent of μ is orthogonal to μ .

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Orthogonality to μ

Results

- Constant sequences \Leftrightarrow PNT
- Periodic sequences \Leftrightarrow PNT in arithmetic Progressions
- Quasiperiodic sequences $f(n) = F(\alpha n \bmod 1)$ - Davenport
- Nilsequences - Green and Tao
- Horocycle Flows - Bourgain, Sarnak and Ziegler
- Dynamical systems with discrete spectrum

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Sarnak Conjecture

Definition

A dynamical system is said to be deterministic, if its topological entropy is 0.

Conjecture (Sarnak conjecture, 2010)

Every bounded complex sequence $\mathbf{u} = (u_n)_{n>0}$ that is obtained by a deterministic dynamical system is orthogonal to the Möbius function $\mu(n)$.

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Automatic Sequences

Definition

Let E be a finite set and σ a k -uniform morphism such that $\sigma(E) \subseteq E^k$. Then if \mathbf{w} is a fixed point of σ , i.e. $\sigma(\mathbf{w}) = \mathbf{w}$, then \mathbf{w} is a k -automatic sequence.

Example (Thue-Morse)

$$E = \{0, 1\}$$

$$\sigma(0) = 01$$

$$\sigma(1) = 10$$

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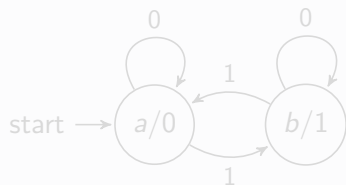
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Deterministic Finite Automata

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \dots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)



$$n = 22 = (10110)_2, \quad u_{22} = 1$$

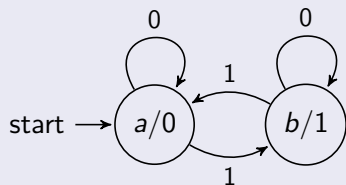
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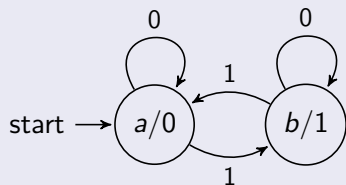
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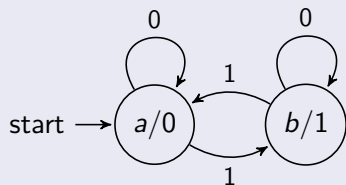
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Every automatic sequence $(a_n)_{n \geq 0}$ fulfills the Sarnak Conjecture

Theorem 2 (M., 2016)

Under suitable (weak) conditions one also gets a Prime Number Theorem for automatic sequence.

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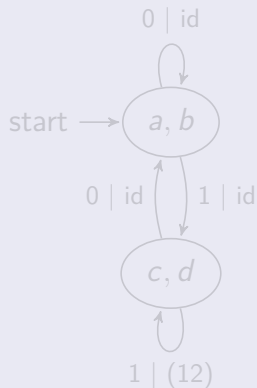
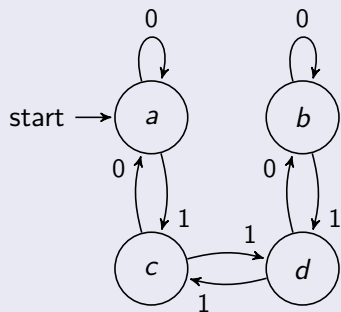
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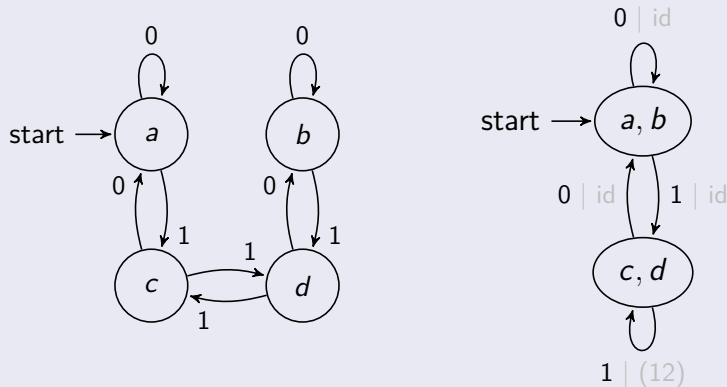
Representation of automatic sequences

Example (Rudin-Shapiro)



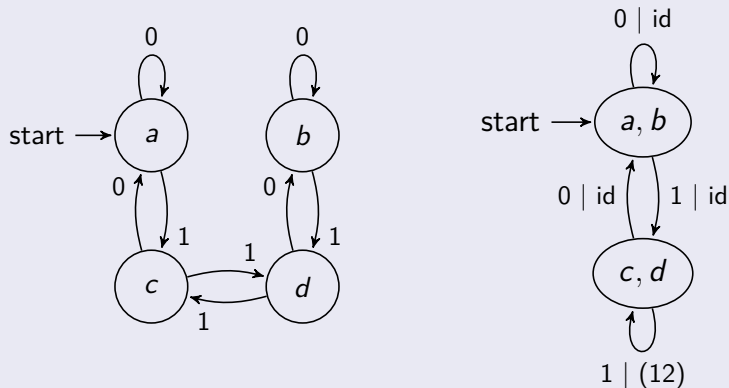
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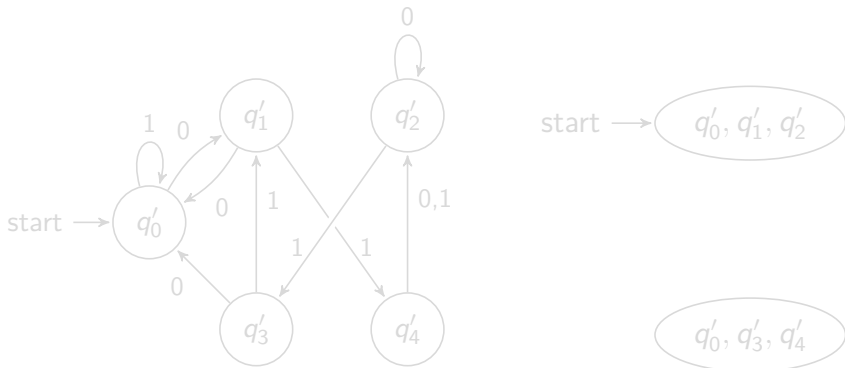
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For every strongly connected automaton A , there exists a naturally induced transducer \mathcal{T}_A . All other naturally induced transducers can be obtained by changing the order on the elements of Q .

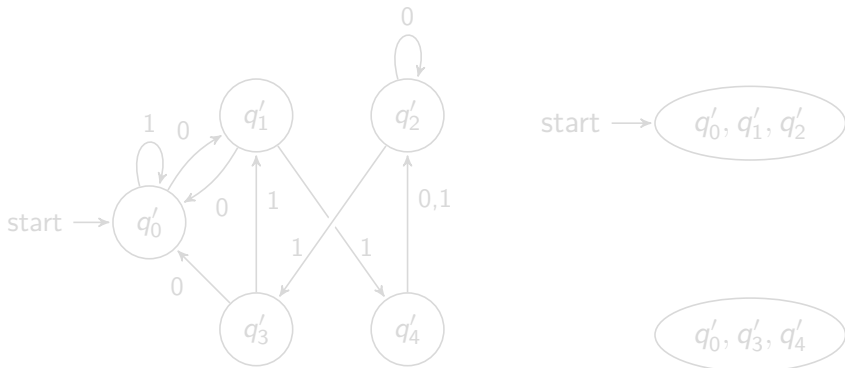
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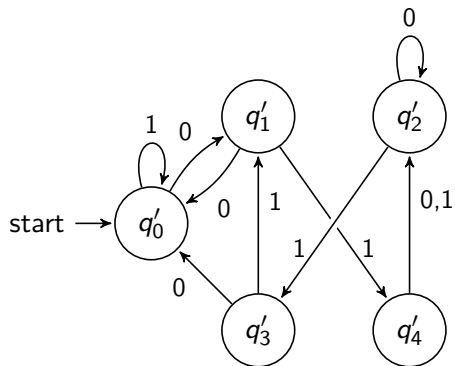
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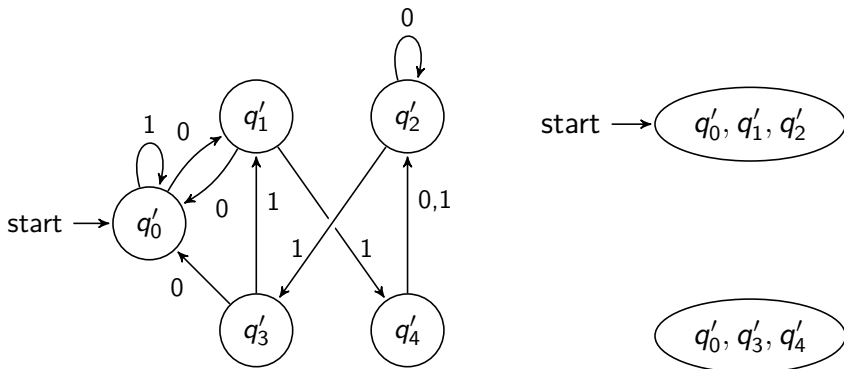
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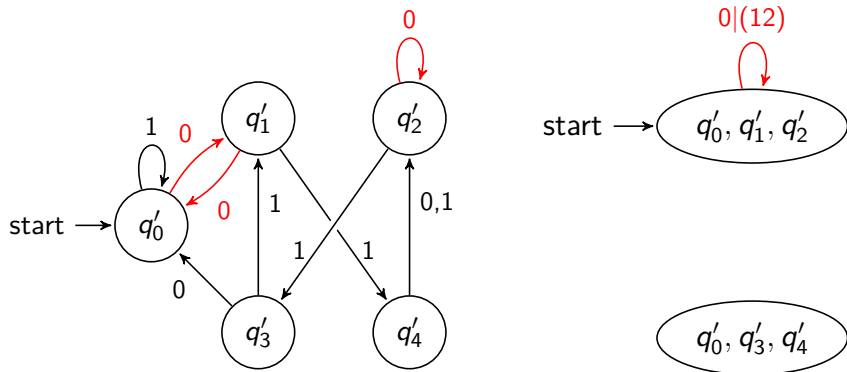
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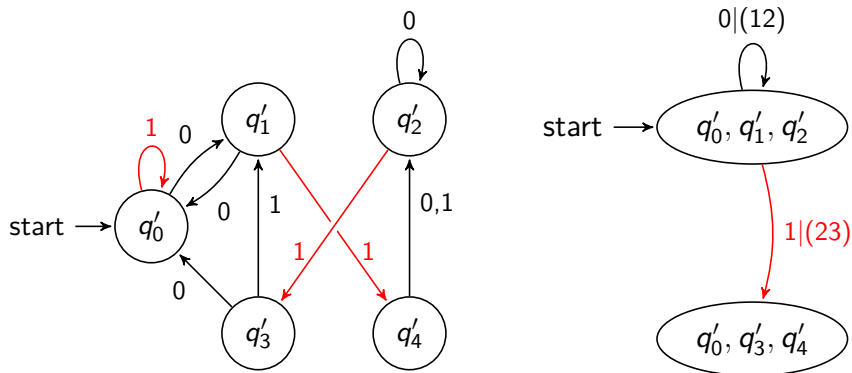
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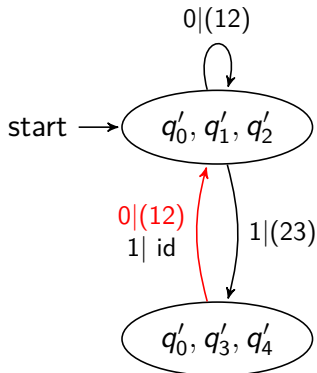
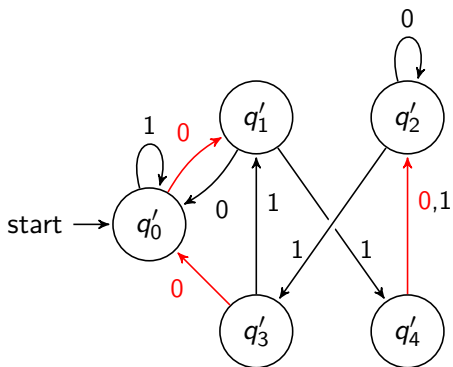
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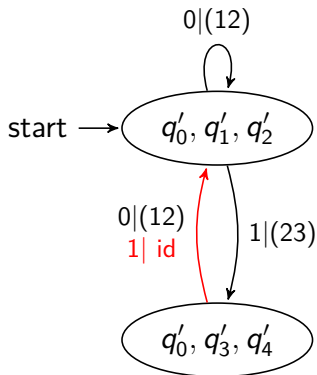
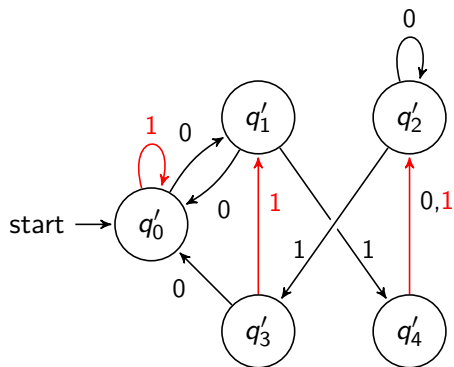
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Techniques

Use and adopt a framework of Mauduit and Rivat developed for the Rudin-Shapiro sequence.

- Carry Property: The contribution of high and low digits is „independent“.
- Fourier Property:
We say that U has the **Fourier property** if there exists $\eta > 0$ and c such that for all λ, α and t

$$\left\| \frac{1}{k^\lambda} \sum_{m < k^\lambda} U(mk^\alpha) e(mt) \right\| \leq ck^{-\eta\lambda}.$$

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Zeckendorf Representation

Fibonacci numbers

$F_0 = 0, F_1 = 1$ and $F_{k+2} = F_{k+1} + F_k$ for $k \geq 0$.

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}},$$

where, φ is the golden ratio.

Zeckendorf Representation

Every positive integer n admits a unique representation

$$n = \sum_{i \geq 2} \varepsilon_i(n) F_i,$$

where, $\varepsilon_i(n) \in \{0, 1\}$ and $\varepsilon_i = 1 \Rightarrow \varepsilon_{i+1} = 0$.

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Zeckendorf sum-of-digits Function

Definition

We denote by

$$s_{\varphi}(n) = \sum_{i \geq 2} \varepsilon_i(n)$$

the Zeckendorf sum-of-digits function.

We note that $s_{\varphi}(n)$ is the least k such that n is the sum of k Fibonacci numbers.

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Results II

Theorem (Drmota, M., Spiegelhofer, 2017)

Let $s_\varphi(n)$ be the Zeckendorf sum-of-digits function and $m(n)$ a bounded multiplicative function. Then we have

$$\sum_{n < N} (-1)^{s_\varphi(n)} m(n) = o(N) \quad (N \rightarrow \infty).$$

This implies that the Zeckendorf sum-of-digits function is orthogonal to the Möbius function.

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Fixpoint of a Substitution

A Morphism

$$a \mapsto ab$$

$$b \mapsto c$$

$$c \mapsto cd$$

$$d \mapsto a.$$

This gives the sequence $(-1)^{s_\varphi(n)}$ under the coding $\tau(a) = \tau(d) = 1, \tau(b) = \tau(c) = -1$.

This is one of the first examples of a substitution with non-constant length to be orthogonal to the Möbius function.

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Sketch of the Proof

- Starting point: (Kátai Criterion) it is sufficient to show

$$\sum_{n \leq N} (-1)^{s_\varphi(pn) + s_\varphi(qn)} = o(N),$$

for all different primes p, q .

- Use a generating function approach and “quasi-additivity” (Carry Property)
- This reduces to find any cancellation in the sum

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Kátai Criterion

Suppose that (x_n) is a bounded complex valued sequence with values in a finite set and that for every pair (p, q) of different prime numbers we have

$$\sum_{n \leq N} x_{pn} \overline{x_{qn}} = o(N).$$

Then for all bounded multiplicative functions $m(n)$ it follows that

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Quasi Additivity

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We say that n_1 and n_2 are r -separated at position k if $\varepsilon_i(n_1) = 0$ for $i \geq k - r$ and $\varepsilon_i(n_2) = 0$ for $i \leq k + r$.

Example:

$$n_1 = 4 \Rightarrow 0000101$$

$$n_2 = 29 \Rightarrow 1010000$$

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Definition (for integer base by Kropf, Wagner)

We call a function $f(n)$ quasi-additive (with respect to the Zeckendorf expansion) if there exists $r \geq 0$ such that



$$f(n_1 + n_2) = f(n_1) + f(n_2)$$

for all integers n_1, n_2 that are r separated.

- $f(n_1) = f(n_2)$ if the Zeckendorf expansion of n_1 and n_2 coincide up to „shifts“.

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We call a function $f(n)$ quasi-additive (with respect to the Zeckendorf expansion) if there exists $r \geq 0$ such that

-

$$f(n_1 + n_2) = f(n_1) + f(n_2)$$

for all integers n_1, n_2 that are r separated.

- $f(n_1) = f(n_2)$ if the Zeckendorf expansion of n_1 and n_2 coincide up to „shifts“.

Common useful properties

- Some kind of Carry Property
 - Carry Property
 - Quasi Additivity
- Recursive Structure
 - Recursion for Fourier Transforms
 - Generating function approach
- Find some saving
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 - Cancellation in the sum

$$\sum_{n < N} (-1)^{s_\varphi(pn) + s_\varphi(qn)}$$

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