The ergodic theory of Schneider's continued fraction map

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Euclidean Algorithm and Gauss Map

By Euclidean algorithm, any rational number $a/b > 1$ can be expressed as

$$x = \frac{a}{b} = a_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots \, c_{n-1} + \frac{1}{c_n}}}},$$

where $c_0, \ldots, c_n$ are natural numbers with $c_n > 1$, except for $n = 0$. Note $c_n(x) = c_{n-1}(Tx)$ for $n \geq 1$, where

$$Tx = \begin{cases} \left\{ \frac{1}{x} \right\} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0, \end{cases}$$

is the famous Gauss map circa 1800.

Regular Continued fraction Expansions

For arbitrary real x we have the regular continued fraction expansion of a real number

$$x = [c_0; c_1, c_2, \dots] = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{1}{c_4 \ddots}}}}.$$

Again $c_n(x) = c_{n-1}(Tx)$ for $n \geq 1$. The terms $c_0, c_1, \cdots$ are called the partial quotients of the continued fraction expansion and the sequence of rational truncates

$$[c_0; c_1, \cdots, c_n] = \frac{p_n}{q_n}, \qquad (n = 1, 2, \cdots)$$

are called the convergents of the continued fraction expansion.

Continued fraction map on [1, 0)

The *Gauss map* $G : [0,1] \to [0,1]$ is the following map:

$$G(x) = \begin{cases} 0 & \text{if } x = 0 \\ \left\{\frac{1}{x}\right\} = \frac{1}{x} \mod 1 & \text{if } 0 < x \leq 1 \end{cases}$$

Here $\{x\}$ denotes the *fractional part* of x. We can write $\{x\} = x - [x]$ where $[x]$ is the integer part. Equivalently, $\{x\} = x \mod 1$.

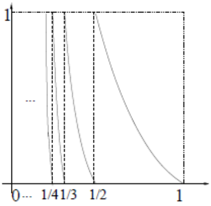
Remark that

$$\left[\frac{1}{x}\right] = n \quad \Leftrightarrow \quad n \leq \frac{1}{x} < n+1 \quad \Leftrightarrow \quad \frac{1}{n+1} < x \leq \frac{1}{n}.$$

Thus, explicitely, one has the following expression (see the graph in Figure 1.1):

$$G(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} - n & \text{if } \frac{1}{n+1} < x \leq \frac{1}{n} \end{cases} \qquad \text{for } n \in \mathbb{N}.$$

The rescrition of G to an interval of the form $(1/n+1, 1/n]$ is called *branch*. Each *branch* $G : (1/n+1, 1/n] \to [0,1)$ is monotone, surjective (onto $[0,1)$) and invertible (see Figure 1.1).



p-adic numbers

Let p be a prime. Any nonzero rational number a can be written in the form $a = p^{\alpha}(r/s)$ where $\alpha \in \mathbb{Z}, r, s \in \mathbb{Z}$ and $p \nmid r, p \nmid s$.

Definition

The *p*-**adic absolute value** of $a \in \mathbb{Q}$ is defined by

$$|a|_p = p^{-\alpha} \quad \text{and} \quad |0|_p = 0.$$

$\mathbb{Q}_p$ is constructed by completing $\mathbb{Q}$ w.r.t. p-adic absolute value.

The p-adic absolute value $|.|_p$ satisfies the following **properties**:

1. $|a|_p = 0$ if and only if $a = 0$,
2. $|ab|_p = |a|_p|b|_p$ for all $a, b \in \mathbb{Q}_p$,
3. $|a+b|_p \leq |a|_p + |b|_p$ for all $a, b \in \mathbb{Q}_p$,
4. $|a+b|_p \leq \max\{|a|_p, |b|_p\}$ for all $a, b \in \mathbb{Q}_p$.

The p-adic absolute value is *non-archimedian*.

p-adic continued fraction expansion

Let p be a prime. We will consider the continued fraction expansion of a p-adic integer $x \in p\mathbb{Z}_p$ in the form

$$x = \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{b_3 + \ddots}}} \tag{1}$$

where $b_n \in \{1, 2, \ldots, p-1\}$, $a_n \in \mathbb{N}$ for $n = 1, 2, \ldots$.

p-adic continued fraction map

For $x \in p\mathbb{Z}_p$ define the map $T_p : p\mathbb{Z}_p \to p\mathbb{Z}_p$ to be

$$T_p(x) = \frac{p^{v(x)}}{x} - \left(\frac{p^{v(x)}}{x} \bmod p \right) = \frac{p^{a(x)}}{x} - b(x) \qquad (2)$$

where $v(x)$ is the p-adic valuation of x, $a(x) \in \mathbb{N}$ and $b(x) \in \{1, 2, \dots, p-1\}$.

We will consider the dynamical system $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ where $\mathcal{B}$ is σ-algebra on $p\mathbb{Z}_p$ and μ is Haar measure on $p\mathbb{Z}_p$.
For the Haar measure it holds $\mu(pa + p^m\mathbb{Z}_p) = p^{1-m}$.

Properties of the p-adic continued fraction map

The following properties are due to Hirsch and Washington (2011).

- T_p is measure-preserving with respect to μ, i.e. $\mu(T_p^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$.
- T_p is ergodic, i.e. $\mu(B) = 0$ or 1 for any $B \in \mathcal{B}$ with $T_p^{-1}(B) = B$.
- The p-adic analogue of Khinchin's Theorem: For almost all $x \in p\mathbb{Z}_p$ the p-adic continued fraction expansion (4) satisfies

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{p}{p-1}.$$

Other properties of the p-adic continued fraction map

Definition

Let T be a measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$. The transformation T is *exact* if

$$\bigcap_{n=0}^{\infty} T^{-n}\mathcal{B} = \mathcal{N}.$$

where $\mathcal{N} = \{B \in \mathcal{B} \,|\, B = \emptyset \text{ a.e. or } B = X \text{ a.e.}\}$.

Theorem (Hančl, Nair, Lertchoosakul, Jaššová)

The p-adic continued fraction map T_p is exact.

Other properties of the p-adic continued fraction map

Because $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ is exact, it implies other strictly weaker properties:

- T_p is strong-mixing, i.e. for all $A, B \in \mathcal{B}$ we have

$$\lim_{n\to\infty} \mu(T_p^{-n} A \cap B) = \mu(A)\mu(B)$$

  which impies

- T_p is weak-mixing, i.e. for all $A, B \in \mathcal{B}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |\mu(T_p^{-j} A \cap B) - \mu(A)\mu(B)| = 0$$

  which implies

- T_p is ergodic, i.e. $\mu(B) = 0$ or 1 for any $B \in \mathcal{B}$ with $T_p^{-1}(B) = B$.

Good Universality

- A sequence of integers $(a_n)_{n=1}^{\infty}$ is called *L^p-good universal* if for each dynamical system $(X, \mathcal{B}, \mu, T)$ and $f \in L^p(X, \mathcal{B}, \mu)$ we have

$$\overline{f}(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_n} x)$$

existing μ almost everywhere.
- A sequence of real numbers $(x_n)_{n=1}^{\infty}$ is *uniformly distributed modulo one* if for each interval $I \subseteq [0, 1)$, if $|I|$ denotes its length, we have

$$\lim_{N \to \infty} \frac{1}{N} \#\{n \le N : \{x_n\} \in I\} = |I|.$$

Subsequence ergodic theory

Lemma

If $(\{a_n\gamma\})_{n=1}^{\infty}$ *is uniformly distributed modulo one for each irrational number* γ*, the dynamical system* $(X, \mathcal{B}, \mu, T)$ *is weak-mixing and* $(a_n)_{n\geq 1}$ *is* L^2*-good universal then* $\overline{f}(x)$ *exists and*

$$\overline{f}(x) = \int_X f d\mu$$

μ *almost everywhere.*

Results

Theorem (Hančl, Nair, Lertchoosakul, Jaššová)

For any L^p-good universal sequence $(k_n)_{n\geq 1}$ where $(\{k_n\gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo one for each irrational number γ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_{k_n} = \frac{p}{p-1},$$

and

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} b_{k_n} = \frac{p}{2},$$

almost everywhere with respect to Haar measure on $p\mathbb{Z}_p$.

Results

Theorem (Hančl, Nair, Lertchoosakul, Jaššová)

For any L^p-good universal sequence $(k_n)_{n\geq 1}$ where $(\{k_n\gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo one for each irrational number γ we have

$$\lim_{N\to\infty}\frac{1}{N}\#\{1\leq n\leq N : a_{k_n}=i\}=\frac{p-1}{p^i};$$

$$\lim_{N\to\infty}\frac{1}{N}\#\{1\leq n\leq N : a_{k_n}\geq i\}=\frac{1}{p^{i-1}};$$

$$\lim_{N\to\infty}\frac{1}{N}\#\{1\leq n\leq N : i\leq a_{k_n}< j\}=\frac{1}{p^{i-1}}\left(1-\frac{1}{p^j}\right);$$

almost everywhere with respect to Haar measure on $p\mathbb{Z}_p$.

Partitions

Let $(X, \mathcal{A}, m)$ be a probability space where X is a set, $\mathcal{A}$ is a σ-algebra of its subsets and m is a probability measure. A partition of $(X, \mathcal{A}, m)$ is defined as a denumerable collection of elements $\alpha = \{A_1, A_2, \dots\}$ of $\mathcal{A}$ such that $A_i \cap A_j = \emptyset$ for all $i, j \in I, i \neq j$ and $\bigcup_{i \in I} A_i = X$. For a measure-preserving transformation T we have $T^{-1}\alpha = \{T^{-1}A_i | A_i \in \alpha\}$ is also a denumerable partition. Given partitions $\alpha = \{A_1, A_2, \dots\}$ and $\beta = \{B_1, B_2, \dots\}$ we define the join of α and β to be the partition $\alpha \vee \beta = \{A_i \cap B_j | A_i \in \alpha, B_j \in \beta\}$.

Entropy of a Partition

For a finite partition $\alpha = \{A_1, \ldots, A_n\}$ we define its entropy $H(\alpha) = -\sum_{i=1}^{n} m(A_i) \log m(A_i)$.

Let $\mathcal{A}' \subset \mathcal{A}$ be a sub-σ-algebra. Then we define the conditional entropy of α given $\mathcal{A}'$ to be $H(\alpha|\mathcal{A}') = -\sum_{i=1}^{n} m(A_i|\mathcal{A}') \log m(A_i|\mathcal{A}')$.

Here of course $m(A|\mathcal{A}')$ means $\mathbb{E}(\chi_A|\mathcal{A}')$ where $\mathbb{E}(.|\mathcal{A}')$ denotes the projection operator $L^1(X, \mathcal{A}, m) \to L^1(X, \mathcal{A}', m)$ and χ_A is the characteristic function of the set A.

Entropy of a transformation

The entropy of a measure-preserving transformation T relative to a partition α is defined to be

$$h_m(T, \alpha) = \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha \right)$$

where the limit always exists. The alternative formula for $h_m(T, \alpha)$ which is used for calculating entropy is

$$h_m(T, \alpha) = \lim_{n \to \infty} H\left(\alpha | \bigvee_{i=1}^{n} T^{-i}\alpha \right) = H\left(\alpha | \bigvee_{i=1}^{\infty} T^{-i}\alpha \right). \quad (3)$$

We define the measure-theoretic entropy of T with respect to the measure m (irrespective of α) to be $h_m(T) = \sup_\alpha h_m(T, \alpha)$ where the supremum is taken over all finite or countable partitions α from $\mathcal{A}$ with $H(\alpha) < \infty$.

Theorem (Jaššová,Nair)

Let $\mathcal{B}$ denote the Haar σ-algebra restricted to $p\mathbb{Z}_p$ and let μ denote Haar measure on $p\mathbb{Z}_p$. Then the measure-preserving transformation $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ has measure-theoretic entropy $\frac{p}{p-1} \log p$.

Isomorphism of measure preserving transformations

Two measure-preseving transformations (X_1, β_1, m_1, T_1) and (X_2, β_2, m_2, T_2) are said to be isomorphic if there exist sets $M_1 \subseteq X_1$ and $M_2 \subseteq X_2$ with $m_1(M_1) = 1$ and $m_2(M_2) = 1$ such that $T_1(M_1) \subseteq M_1$ and $T_2(M_2) \subseteq M_2$ and such that there exists a map $\phi : M_1 \to M_2$ satisfying $\phi T_1(x) = T_2\phi(x)$ for all $x \in M_1$ and $m_1(\phi^{-1}(A)) = m_2(A)$ for all $A \in \beta_2$. The importance of measure theoretic entropy, is that two dynamical systems with different entropies can not be isomorphic.

Bernoulli Space

Suppose (Y, α, l) is a probability space, and let $(X, \beta, m) = \Pi_{-\infty}^{\infty}(Y, \alpha, l)$ i.e. the bi-infinite product probability space. For shift map $\tau(\{x_n\}) = (\{x_{n+1}\})$, the measure preserving transformation (X, β, m, τ) is called the Bernoulli process with state space (Y, α, l). Here $\{x_n\}$ is a bi-infinite sequence of elements of the set Y. Any measure preserving transformation isomorphic to a Bernoulli process will be refered to as Bernoulli.

Ornstein's theorem

The fundamental fact about Bernoulli processes, famously proved by D. Ornstein in 1970, is that Bernoulli processes with the same entropy are isomorphic.

The natural extention

To any measure-preserving transformation, (X, β, m, T_0) set $X^\infty = \Pi_{n=0}^{\infty} X$ and set

$$X_{T_0} = \{\underline{x} = (x_0, x_1, \dots) \in X^\infty : x_n = T_0(x_{n+1}), x_n \in X, n = 0, 1, 2, \dots\},$$

and let $T : X_{T_0} \to X_{T_0}$ be defined by

$$T((x_0, x_1, \dots,)) = (T(x_0), x_0, x_1, \dots,).$$

The map T is bijective on X_{T_0}. If T_0 preserves a measure m, then we can define a measure $\overline{m}$ on X_{T_0}, by defining $\overline{m}$ on the cylinder sets $C(A_0, A_1, \dots, A_k) = \{\underline{x} : x_0 \in A_0, x_1 \in A_1, \dots, x_k \in A_k\}$ by

$$\overline{m}(C(A_0, A_1, \dots, A_k)) = m(T_0^{-k}(A_0) \cap T_0^{-k+1}(A_1) \cap \dots \cap A_k),$$

for $k \geq 1$. One can check that the invertable transformation $(X_{T_0}, \overline{\beta}, \overline{m}, T_0)$ called the natural extention of (X, β, m, T_0) is measure preserving as a consequence of the measure preservation of the transformation (X, β, m, T_0).

Fundamental Dynamical property of the Schneider Map

Theorem (Jaššová,Nair)

Suppose $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ is the Schneider continued fraction map. Then the dynamical system $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ has a natural extention that is Bernoulli.

This property implies all the mixing properties of the map and via ergodic theorems all the properties of averages of convergents. Also, via Ornstein's theorem, it is isomphorphic as a dynamical system to all Bernoulli shifts with the same entropy and hence is completely classified.

Absolute values on topological fields

Let K denote a topological field. By this we mean that the field K is a locally compact group under the addition , with respect a topology (which in our case is discrete). This ensures that K comes with a translation invariant Haar measure μ on K, that is unique up to scalar multiplication. For an element $a \in K$, we are now able it absolute value, as

$$|a| = \frac{\mu(aX)}{\mu(X)},$$

for every μ masureable $X \subseteq K$ of finite μ measure. An absolute value is a function $|.| : K \to \mathbb{R}_{\geq 0}$ such that (i) $|a| = 0$ if and only if $a = 0$; (ii) $|ab| = |a||b|$ for all $a, b \in K$ and $|a + b| \leq |a| + |b|$ for all pairs $a, b \in K$. The absolute value just defined gives rise to a metric defined by $d(a, b) = |a - b|$ with $a, b \in K$, whose topology coincides with original topology on the field K.

Archemedian and Non-Archemedian

Topological fields come in two types. The first where (iii) can be replace by the stronger condition (iii)*
$|a + b| \leq \max(|a|, |b|)$ $a, b \in K$, called non-archimedean spaces and spaces where (iii)* is not true called archimedean spaces. fields. In this paper we shall concern ourselves solely with non-archimedean fields.

Valuations and absolute values

Another approach to defining a non-archimedan field is via discrete valuations. Let $K^* = K \backslash \{0\}$. A map $v : K^* \to \mathbb{R}$ is a valuation if (i) $v(K^*) \neq \{0\}$; (ii) $v(xy) = v(x) + v(y)$ for $x, y \in K$ and (iii) $v(x+y) \geq \min\{v(x), v(y)\}$. Two valuations v and cv, for $c > 0$ a real constant, are called equivalent. A valuation determines a non-trivial non-Archimedean absolute value and vice versa. We extend v to K formally by letting $v(0) = 1$. The image $v(K^*)$ is an additive subgroup of $\mathbb{R}$, the value group of v. If it is discrete, i.e., isomorphic to $\mathbb{Z}$, we say v is a discrete valuation. If $v(K^*) = \mathbb{Z}$, we call v normalised discrete valuation. To our initial valuation we associate the valuation described as follows. Pick $0 < \alpha < 1$ and write $|a| = \alpha^{v(a)}$, i.e., let $v(a) = \log_\alpha |a|$. Then $v(a)$ is a valuation, an additive version of $|a|$.

Rings of integers and maximal ideals

Let $v : K^* \to \mathbb{R}$ be a valuation corresponding to the absolute value $|.| : K \to \mathbb{R}_{\geq 0}$. Then

$$\mathcal{O} = \mathcal{O}_v := \{x \in K : v(x) \geq 0\} = \mathcal{O}_K := \{x \in K : |x| \leq 1\}$$

is a ring, called the valuation ring of v. and K is its field of fractions, and if $x \in K \backslash \mathcal{O}$ then $\frac{1}{x} \in \mathcal{O}$. The set of units in $\mathcal{O}$ is $\mathcal{O}^\times = \{x \in K : v(x) = 0\} = \{x \in K : |x| = 1\}$ and $\mathcal{M} = \{x \in K : v(x) > 0\} = \{x \in K : |x| < 1\}$ is an ideal in $\mathcal{O}$. Because $\mathcal{O} = \mathcal{O}^\times \cup \mathcal{M}$, is a unique maximal ideal, so $\mathcal{O}$ is local and $k = \mathcal{O}/\mathcal{M}$ is a field, called the residue field of v or of K.

The structure of maximal ideals

In the sequel throughout these lectures we assume that k is a finite field. Suppose $v : K^* \to \mathbb{Z}$ is normalised discrete. Take $\pi \in \mathcal{M}$ such that $v(\pi) = 1$. We call π a uniformiser. Then every $x \in K$ can be written uniquely as $x = u\pi^n$ with $u \in \mathcal{O}^\times$ and $n \in \mathbb{Z}_{\geq 0}$. Also every $x \in \mathcal{M}$ can be written uniquely as $x = u\pi^n$ for a unit $u \in \mathcal{O}^\times$ and $n \geq 1$. In particular, $\mathcal{M} = (\pi^n)$ is a principal ideal . Moreover, every ideal $I \subset \mathcal{O}$ is principal, as $(0) \neq I \subset \mathcal{O}$ implies $I = (\pi^n)$ where $n = \min\{v(x) : x \in I\}$, so $\mathcal{O}$ is a principal ideal domain (PID).

There are two examples

(i) The p-adic numbers $\mathbb{Q}_p$ and their finite extentions. For instance if $K = \mathbb{Q}_p$ then $\mathcal{O} = \mathbb{Z}_p$ $\mathcal{M} = p\mathbb{Z}_p$. Here we can take $\pi = p$.
(ii) The field of formal power series $K = \mathbb{F}_q((X^{-1}))$ for $q = p^n$ for some prime p, with $\mathcal{O} = \mathbb{F}_q[X]$ and $\mathcal{M} = I(x)\mathbb{F}_q[X]$ for some irreducible polynomial I. Here we can take $\pi - I$.
These two are the only two possibilities. This is the structure theorem for non-archemedian fields.

Schneider's Map on an arbitrary non-archemedean field

We define the map $T_v : \mathcal{M} \to \mathcal{M}$ defined by

$$T_v(x) = \frac{\pi^{v(x)}}{x} - b(x)$$

where $b(x)$ denotes the residue class to which $\frac{\pi^{v(x)}}{x}$ in k.
This gives rise to the continued fraction expansion of $x \in \mathcal{M}$ in the form

$$x = \cfrac{\pi^{a_1}}{b_1 + \cfrac{\pi^{a_2}}{b_2 + \cfrac{\pi^{a_3}}{b_3 + \ddots}}} \tag{4}$$

where $b_n \in k^\times, a_n \in \mathbb{N}$ for $n = 1, 2, \dots$.

The start of continued fractions on a non-archemedean field

The rational approximants to $x \in \mathcal{M}$ arise in a manner similar to that in the case of the real numbers as follows. We suppose $A_0 = b_0, B_0 = 1, A_1 = b_0 b_1 + \pi^{a_1}, B_1 = b_1$. Then set

$$A_n = \pi^{a_n} A_{n-2} + b_n A_{n-1} \text{ and } B_n = \pi^{a_n} B_{n-2} + b_n B_{n-1} \quad (5)$$

for $n \geq 2$. A simple inductive argument gives for $n = 1, 2, \ldots$.

$$A_{n-1} B_n - A_n B_{n-1} = (-1)^n \pi^{a_1 + \ldots + a_n}. \quad (6)$$

Dynamics of the Schneider's map on a non-archemedean field

The map $T_v : \mathcal{M} \to \mathcal{M}$ preserves Haar measure on $\mathcal{M}$. We also have the following.

Theorem

Let $\mathcal{B}$ denote the Haar σ-algebra restricted to $\mathcal{M}$ and let μ denote Haar measure on $\mathcal{M}$. Then the measure-preserving transformation $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ has measure-theoretic entropy $\frac{|k|}{|k^\times|}\log(|k|)$.

Theorem

Suppose $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ is as in our first theorem. Then the dynamical system $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ has a natural extension that is Bernoulli.

This tells us the isomorphism class of the dynamical system $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ is determined by its residue class field irrespective of the characteristic. This means for different p each Sneider map on the p-adics non-isomorphic.

Results

Theorem (Nair,Jaššová)

For any L^p-good universal sequence $(k_n)_{n\geq 1}$ where $(\{k_n\gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo one for each irrational number γ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}a_{k_n}=\frac{|k|}{|k^{\times}|},$$

and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}b_{k_n}=\frac{|k|}{2},$$

almost everywhere with respect to Haar measure on $\mathcal{M}$.

Results

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For any L^p-good universal sequence $(k_n)_{n\geq 1}$ where $(\{k_n\gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo one for each irrational number γ we have

$$\lim_{N\to\infty} \frac{1}{N}\#\{1 \leq n \leq N : a_{k_n} = i\} = \frac{|k^{\times}|}{|k|^{i}};$$

$$\lim_{N\to\infty} \frac{1}{N}\#\{1 \leq n \leq N : a_{k_n} \geq i\} = \frac{1}{|k|^{i-1}};$$

$$\lim_{N\to\infty} \frac{1}{N}\#\{1 \leq n \leq N : i \leq a_{k_n} < j\} = \frac{1}{|k|^{i-1}}\left(1 - \frac{1}{|k|^{j}}\right);$$

almost everywhere with respect to Haar measure on $\mathcal{M}$.

Thank you for your attention.