The ergodic theory of Schneider’s continued fraction map

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By Euclidean algorithm, any rational number \( a/b > 1 \) can be expressed as

\[
x = \frac{a}{b} = a_0 + \frac{1}{c_1 + \frac{1}{c_2 + \frac{1}{c_3 + \cdots}}},
\]

where \( c_0, \ldots, c_n \) are natural numbers with \( c_n > 1 \), except for \( n = 0 \). Note \( c_n(x) = c_{n-1}(Tx) \) for \( n \geq 1 \), where

\[
Tx = \begin{cases} 
\{ \frac{1}{x} \} & \text{if } x \neq 0; \\
0 & \text{if } x = 0,
\end{cases}
\]

is the famous Gauss map circa 1800.
Regular Continued fraction Expansions

For arbitrary real $x$ we have the regular continued fraction expansion of a real number

$$x = [c_0; c_1, c_2, \ldots] = c_0 + \frac{1}{c_1 + \frac{1}{c_2 + \frac{1}{c_3 + \frac{1}{c_4 + \ddots}}}}.$$

Again $c_n(x) = c_{n-1}(Tx)$ for $n \geq 1$. The terms $c_0, c_1, \cdots$ are called the partial quotients of the continued fraction expansion and the sequence of rational truncates

$$[c_0; c_1, \cdots, c_n] = \frac{p_n}{q_n}, \quad (n = 1, 2, \cdots)$$

are called the convergents of the continued fraction expansion.
Continued fraction map on \([1, 0)\)

The Gauss map \(G : [0, 1) \rightarrow [0, 1)\) is the following map:

\[
G(x) = \begin{cases} 
0 & \text{if } x = 0 \\
\{ \frac{1}{\lfloor x \rfloor} \} = \frac{1}{\lfloor x \rfloor} \mod 1 & \text{if } 0 < x \leq 1
\end{cases}
\]

Here \(\{x\}\) denotes the fractional part of \(x\). We can write \(\{x\} = x - \lfloor x \rfloor\) where \(\lfloor x \rfloor\) is the integer part. Equivalently, \(\{x\} = x \mod 1\).

Remark that

\[
\frac{1}{\lfloor x \rfloor} = n \iff n \leq \frac{1}{x} < n + 1 \iff \frac{1}{n + 1} < x \leq \frac{1}{n}.
\]

Thus, explicitly, one has the following expression (see the graph in Figure 1.1):

\[
G(x) = \begin{cases} 
0 & \text{if } x = 0 \\
\frac{1}{\lfloor x \rfloor} - n & \text{if } \frac{1}{n+1} < x \leq \frac{1}{n} \quad \text{for } n \in \mathbb{N}
\end{cases}
\]

The restriction of \(G\) to an interval of the form \((1/n + 1, 1/n]\) is called branch. Each branch \(G : (1/n + 1, 1/n] \rightarrow [0, 1)\) is monotone, surjective (onto \([0, 1)\)) and invertible (see Figure 1.1).
**$p$-adic numbers**

Let $p$ be a prime. Any nonzero rational number $a$ can be written in the form $a = p^\alpha(r/s)$ where $\alpha \in \mathbb{Z}$, $r, s \in \mathbb{Z}$ and $p \nmid r, p \nmid s$.

**Definition**

The **$p$-adic absolute value** of $a \in \mathbb{Q}$ is defined by

$$|a|_p = p^{-\alpha} \quad \text{and} \quad |0|_p = 0.$$ 

$\mathbb{Q}_p$ is constructed by completing $\mathbb{Q}$ w.r.t. $p$-adic absolute value.

The $p$-adic absolute value $|.|_p$ satisfies the following **properties**:

1. $|a|_p = 0$ if and only if $a = 0$,
2. $|ab|_p = |a|_p|b|_p$ for all $a, b \in \mathbb{Q}_p$,
3. $|a + b|_p \leq |a|_p + |b|_p$ for all $a, b \in \mathbb{Q}_p$,
4. $|a + b|_p \leq \max\{|a|_p, |b|_p\}$ for all $a, b \in \mathbb{Q}_p$.

The $p$-adic absolute value is **non-archimedean**.
Let $p$ be a prime. We will consider the continued fraction expansion of a $p$-adic integer $x \in p\mathbb{Z}_p$ in the form

$$x = \frac{p^{a_1}}{b_1 + \frac{p^{a_2}}{b_2 + \frac{p^{a_3}}{b_3 + \ldots}}}$$

where $b_n \in \{1, 2, \ldots, p - 1\}$, $a_n \in \mathbb{N}$ for $n = 1, 2, \ldots$. 
\( p \)-adic continued fraction map

For \( x \in p\mathbb{Z}_p \) define the map \( T_p : p\mathbb{Z}_p \to p\mathbb{Z}_p \) to be

\[
T_p(x) = \frac{p^{v(x)}}{x} - \left( \frac{p^{v(x)}}{x} \mod p \right) = \frac{p^{a(x)}}{x} - b(x)
\]

(2)

where \( v(x) \) is the \( p \)-adic valuation of \( x \), \( a(x) \in \mathbb{N} \) and \( b(x) \in \{1, 2, \ldots , p - 1\} \).

We will consider the dynamical system \((p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)\) where \( \mathcal{B} \) is \( \sigma \)-algebra on \( p\mathbb{Z}_p \) and \( \mu \) is Haar measure on \( p\mathbb{Z}_p \).

For the Haar measure it holds \( \mu(p^a + p^m\mathbb{Z}_p) = p^{1-m} \).
Properties of the $p$-adic continued fraction map

The following properties are due to Hirsch and Washington (2011).

- $T_p$ is measure-preserving with respect to $\mu$, i.e. $\mu(T_p^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$.
- $T_p$ is ergodic, i.e. $\mu(B) = 0$ or $1$ for any $B \in \mathcal{B}$ with $T_p^{-1}(B) = B$.
- The $p$-adic analogue of Khinchin’s Theorem: For almost all $x \in p\mathbb{Z}_p$ the $p$-adic continued fraction expansion (4) satisfies
  \[
  \lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{p}{p - 1}.
  \]
Other properties of the $p$-adic continued fraction map

Definition
Let $T$ be a measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$. The transformation $T$ is exact if

$$\bigcap_{n=0}^{\infty} T^{-n}\mathcal{B} = \mathcal{N},$$

where $\mathcal{N} = \{ B \in \mathcal{B} \mid B = \emptyset \text{ a.e. or } B = X \text{ a.e.} \}.$

Theorem (Hančl, Nair, Lertchoosakul, Jaššová)

The $p$-adic continued fraction map $T_p$ is exact.
Other properties of the $p$-adic continued fraction map

Because $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ is exact, it implies other strictly weaker properties:

- $T_p$ is strong-mixing, i.e. for all $A, B \in \mathcal{B}$ we have
  \[
  \lim_{n \to \infty} \mu(T_p^{-n}A \cap B) = \mu(A)\mu(B)
  \]
  which implies

- $T_p$ is weak-mixing, i.e. for all $A, B \in \mathcal{B}$ we have
  \[
  \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} |\mu(T_p^{-j}A \cap B) - \mu(A)\mu(B)| = 0
  \]
  which implies

- $T_p$ is ergodic, i.e. $\mu(B) = 0$ or $1$ for any $B \in \mathcal{B}$ with $T_p^{-1}(B) = B$. 

Good Universality

- A sequence of integers \((a_n)_{n=1}^{\infty}\) is called \(L^p\)-good universal if for each dynamical system \((X, \mathcal{B}, \mu, T)\) and \(f \in L^p(X, \mathcal{B}, \mu)\) we have

  \[
  \bar{f}(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_n}x)
  \]

  existing \(\mu\) almost everywhere.

- A sequence of real numbers \((x_n)_{n=1}^{\infty}\) is uniformly distributed modulo one if for each interval \(I \subseteq [0, 1)\), if \(|I|\) denotes its length, we have

  \[
  \lim_{N \to \infty} \frac{1}{N} \# \{ n \leq N : \{ x_n \} \in I \} = |I|.
  \]
Lemma

If \( \{a_n\gamma\}_{n=1}^{\infty} \) is uniformly distributed modulo one for each irrational number \( \gamma \), the dynamical system \((X, \mathcal{B}, \mu, T)\) is weak-mixing and \((a_n)_{n \geq 1}\) is \(L^2\)-good universal then \( \overline{f}(x) \) exists and

\[
\overline{f}(x) = \int_X f d\mu
\]

\( \mu \) almost everywhere.
Results

Theorem (Hančl, Nair, Lertchoosakul, Jaššová)

For any $L^p$-good universal sequence $(k_n)_{n \geq 1}$ where $(\{k_n \gamma\})_{n=1}^\infty$ is uniformly distributed modulo one for each irrational number $\gamma$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_{k_n} = \frac{p}{p-1},$$

and

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} b_{k_n} = \frac{p}{2},$$

almost everywhere with respect to Haar measure on $p\mathbb{Z}_p$. 
Results

Theorem (Hančeľ, Nair, Lertchoosakul, Jaššová)

For any $L^p$-good universal sequence $(k_n)_{n \geq 1}$ where $(\{k_n \gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo one for each irrational number $\gamma$ we have

$$\lim_{N \to \infty} \frac{1}{N} \# \{1 \leq n \leq N : a_{k_n} = i\} = \frac{p - 1}{p^i};$$

$$\lim_{N \to \infty} \frac{1}{N} \# \{1 \leq n \leq N : a_{k_n} \geq i\} = \frac{1}{p^{i-1}};$$

$$\lim_{N \to \infty} \frac{1}{N} \# \{1 \leq n \leq N : i \leq a_{k_n} < j\} = \frac{1}{p^{i-1}} \left(1 - \frac{1}{p^j}\right);$$

almost everywhere with respect to Haar measure on $p\mathbb{Z}_p$. 
Let $(X, \mathcal{A}, m)$ be a probability space where $X$ is a set, $\mathcal{A}$ is a $\sigma$-algebra of its subsets and $m$ is a probability measure. A partition of $(X, \mathcal{A}, m)$ is defined as a denumerable collection of elements $\alpha = \{A_1, A_2, \ldots\}$ of $\mathcal{A}$ such that $A_i \cap A_j = \emptyset$ for all $i, j \in I, i \neq j$ and $\bigcup_{i \in I} A_i = X$. For a measure-preserving transformation $T$ we have $T^{-1}\alpha = \{T^{-1}A_i|A_i \in \alpha\}$ is also a denumerable partition. Given partitions $\alpha = \{A_1, A_2, \ldots\}$ and $\beta = \{B_1, B_2, \ldots\}$ we define the join of $\alpha$ and $\beta$ to be the partition $\alpha \vee \beta = \{A_i \cap B_j|A_i \in \alpha, B_j \in \beta\}$. 
Entropy of a Partition

For a finite partition $\alpha = \{A_1, \ldots, A_n\}$ we define its entropy
$H(\alpha) = - \sum_{i=1}^{n} m(A_i) \log m(A_i)$.

Let $\mathcal{A}' \subset \mathcal{A}$ be a sub-$\sigma$-algebra. Then we define the conditional
entropy of $\alpha$ given $\mathcal{A}'$ to be
$H(\alpha|\mathcal{A}') = - \sum_{i=1}^{n} m(A_i|\mathcal{A}') \log m(A_i|\mathcal{A}')$.

Here of course $m(A|\mathcal{A}')$ means $\mathbb{E}(\chi_A|\mathcal{A}')$ where $\mathbb{E}(.|\mathcal{A}')$ denotes
the projection operator $L^1(X, \mathcal{A}, m) \to L^1(X, \mathcal{A}', m)$ and $\chi_A$ is the
characteristic function of the set $A$. 
Entropy of a transformation

The entropy of a measure-preserving transformation $T$ relative to a partition $\alpha$ is defined to be

$$h_m(T, \alpha) = \lim_{n \to \infty} \frac{1}{n} H \left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \right)$$

where the limit always exists. The alternative formula for $h_m(T, \alpha)$ which is used for calculating entropy is

$$h_m(T, \alpha) = \lim_{n \to \infty} H \left( \left. \alpha \right| \bigvee_{i=1}^{n} T^{-i} \alpha \right) = H \left( \left. \alpha \right| \bigvee_{i=1}^{\infty} T^{-i} \alpha \right). \quad (3)$$

We define the measure-theoretic entropy of $T$ with respect to the measure $m$ (irrespective of $\alpha$) to be $h_m(T) = \sup_{\alpha} h_m(T, \alpha)$ where the supremum is taken over all finite or countable partitions $\alpha$ from $\mathcal{A}$ with $H(\alpha) < \infty$. 
Theorem (Jaššová, Nair)

Let $\mathcal{B}$ denote the Haar $\sigma$-algebra restricted to $p\mathbb{Z}_p$ and let $\mu$ denote Haar measure on $p\mathbb{Z}_p$. Then the measure-preserving transformation $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ has measure-theoretic entropy $\frac{p}{p-1} \log p$. 
Isomorphism of measure preserving transformations

Two measure-preserving transformations \((X_1, \beta_1, m_1, T_1)\) and \((X_2, \beta_2, m_2, T_2)\) are said to be isomorphic if there exist sets \(M_1 \subseteq X_1\) and \(M_2 \subseteq X_2\) with \(m_1(M_1) = 1\) and \(m_2(M_2) = 1\) such that \(T_1(M_1) \subseteq M_1\) and \(T_2(M_2) \subseteq M_2\) and such that there exists a map \(\phi : M_1 \to M_2\) satisfying \(\phi T_1(x) = T_2\phi(x)\) for all \(x \in M_1\) and \(m_1(\phi^{-1}(A)) = m_2(A)\) for all \(A \in \beta_2\). The importance of measure theoretic entropy, is that two dynamical systems with different entropies can not be isomorphic.
Suppose \((Y, \alpha, \mathcal{I})\) is a probability space, and let 
\((X, \beta, m) = \prod_{-\infty}^{\infty} (Y, \alpha, \mathcal{I})\) i.e. the bi-infinite product probability space. For shift map \(\tau(\{x_n\}) = (\{x_{n+1}\})\), the measure preserving transformation \((X, \beta, m, \tau)\) is called the Bernoulli process with state space \((Y, \alpha, \mathcal{I})\). Here \(\{x_n\}\) is a bi-infinite sequence of elements of the set \(Y\). Any measure preserving transformation isomorphic to a Bernoulli process will be referred to as Bernoulli.
Ornstein’s theorem

The fundamental fact about Bernoulli processes, famously proved by D. Ornstein in 1970, is that Bernoulli processes with the same entropy are isomorphic.
The natural extension

To any measure-preserving transformation, \((X, \beta, m, T_0)\) set \(X^\infty = \prod_{n=0}^\infty X\) and set

\[X_{T_0} = \{\underline{x} = (x_0, x_1, \ldots) \in X^\infty : x_n = T_0(x_{n+1}), x_n \in X, n = 0, 1, 2, \ldots \},\]

and let \(T : X_{T_0} \to X_{T_0}\) be defined by

\[T((x_0, x_1, \ldots, )) = (T(x_0), x_0, x_1, \ldots, ).\]

The map \(T\) is bijective on \(X_{T_0}\). If \(T_0\) preserves a measure \(m\), then we can define a measure \(m\) on \(X_{T_0}\), by defining \(m\) on the cylinder sets \(C(A_0, A_1, \ldots, A_k) = \{\underline{x} : x_0 \in A_0, x_1 \in A_1, \ldots, x_k \in A_k\}\) by

\[m(C(A_0, A_1, \ldots, A_k)) = m(T_0^{-k}(A_0) \cap T_0^{-k+1}(A_1) \cap \ldots \cap A_k),\]

for \(k \geq 1\). One can check that the invertable transformation \((X_{T_0}, \beta, m, T_0)\) called the natural extention of \((X, \beta, m, T_0)\) is measure preserving as a consequence of the measure preservation of the transformation \((X, \beta, m, T_0)\).
Theorem (Jaššová,Nair)

Suppose \((p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)\) is the Schneider continued fraction map. Then the dynamical system \((p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)\) has a natural extension that is Bernoulli.

This property implies all the mixing properties of the map and via ergodic theorems all the properties of averages of convergents. Also, via Ornstein’s theorem, it is isomorphic as a dynamical system to all Bernoulli shifts with the same entropy and hence is completely classified.
Absolute values on topological fields

Let $K$ denote a topological field. By this we mean that the field $K$ is a locally compact group under the addition, with respect a topology (which in our case is discrete). This ensures that $K$ comes with a translation invariant Haar measure $\mu$ on $K$, that is unique up to scalar multiplication. For an element $a \in K$, we are now able it absolute value, as

$$|a| = \frac{\mu(aX)}{\mu(X)},$$

for every $\mu$ measurable $X \subseteq K$ of finite $\mu$ measure. An absolute value is a function $|.| : K \to \mathbb{R}_{\geq 0}$ such that (i) $|a| = 0$ if and only if $a = 0$; (ii) $|ab| = |a||b|$ for all $a, b \in K$ and $|a + b| \leq |a| + |b|$ for all pairs $a, b \in K$. The absolute value just defined gives rise to a metric defined by $d(a, b) = |a - b|$ with $a, b \in K$, whose topology coincides with original topology on the field $K$. 
Topological fields come in two types. The first where (iii) can be replace by the stronger condition (iii)*
\[|a + b| \leq \max(|a|, |b|) \quad a, b \in K, \text{ called non-archimedean spaces}
\]
and spaces where (iii)* is not true called archimedean spaces. fields. In this paper we shall concern ourselves solely with non-archimedean fields.
Another approach to defining a non-archimedan field is via discrete valuations. Let $K^* = K \setminus \{0\}$. A map $v : K^* \to \mathbb{R}$ is a valuation if (i) $v(K^*) \neq \{0\}$; (ii) $v(xy) = v(x) + v(y)$ for $x, y \in K$ and (iii) $v(x + y) \geq \min\{v(x), v(y)\}$. Two valuations $v$ and $cv$, for $c > 0$ a real constant, are called equivalent. A valuation determines a non-trivial non-Archimedean absolute value and vice versa. We extend $v$ to $K$ formally by letting $v(0) = 1$. The image $v(K^*)$ is an additive subgroup of $\mathbb{R}$, the value group of $v$. If it is discrete, i.e., isomorphic to $\mathbb{Z}$, we say $v$ is a discrete valuation. If $v(K^*) = \mathbb{Z}$, we call $v$ normalised discrete valuation. To our initial valuation we associate the valuation described as follows. Pick $0 < \alpha < 1$ and write $|a| = \alpha^{v(a)}$, i.e., let $v(a) = \log_\alpha |a|$. Then $v(a)$ is a valuation, an additive version of $|a|$. 
Rings of integers and maximal ideals

Let $v : K^* \rightarrow \mathbb{R}$ be a valuation corresponding to the absolute value $|.| : K \rightarrow \mathbb{R}_{\geq 0}$. Then

$$\mathcal{O} = \mathcal{O}_v := \{x \in K : v(x) \geq 0\} = \mathcal{O}_K := \{x \in K : |x| \leq 1\}$$

is a ring, called the valuation ring of $v$. and $K$ is its field of fractions, and if $x \in K \setminus \mathcal{O}$ then $\frac{1}{x} \in \mathcal{O}$. The set of units in $\mathcal{O}$ is $\mathcal{O}^\times = \{x \in K : v(x) = 0\} = \{x \in K : |x| = 1\}$ and $\mathcal{M} = \{x \in K : v(x) > 0\} = \{x \in K : |x| < 1\}$ is an ideal in $\mathcal{O}$. Because $\mathcal{O} = \mathcal{O}^\times \cup \mathcal{M}$, is a unique maximal ideal, so $\mathcal{O}$ is local and $k = \mathcal{O}/\mathcal{M}$ is a field, called the residue field of $v$ or of $K$. 
The structure of maximal ideals

In the sequel throughout these lectures we assume that $k$ is a finite field. Suppose $\nu : K^* \to \mathbb{Z}$ is normalised discrete. Take $\pi \in \mathcal{M}$ such that $\nu(\pi) = 1$. We call $\pi$ a uniformiser. Then every $x \in K$ can be written uniquely as $x = u\pi^n$ with $u \in \mathcal{O}^\times$ and $n \in \mathbb{Z}_{\geq 0}$. Also every $x \in \mathcal{M}$ can be written uniquely as $x = u\pi^n$ for a unit $u \in \mathcal{O}^\times$ and $n \geq 1$. In particular, $\mathcal{M} = (\pi^n)$ is a principal ideal. Moreover, every ideal $I \subset \mathcal{O}$ is principal, as $(0) \neq I \subset \mathcal{O}$ implies $I = (\pi^n)$ where $n = \min\{\nu(x) : x \in I\}$, so $\mathcal{O}$ is a principal ideal domain (PID).
There are two examples

(i) The $p$-adic numbers $\mathbb{Q}_p$ and their finite extentions. For instance if $K = \mathbb{Q}_p$ then $\mathcal{O} = \mathbb{Z}_p$ $\mathcal{M} = p\mathbb{Z}_p$. Here we can take $\pi = p$.

(ii) The field of formal power series $K = \mathbb{F}_q((X^{-1}))$ for $q = p^n$ for some prime $p$, with $\mathcal{O} = \mathbb{F}_q[X]$ and $\mathcal{M} = I(x)\mathbb{F}_q[X]$ for some irreducible polynomial $I$. Here we can take $\pi - I$.

These two are the only two possibilities. This is the structure theorem for non-archemedian fields.
We define the map \( T_v : \mathcal{M} \rightarrow \mathcal{M} \) defined by

\[
T_v(x) = \frac{\pi^v(x)}{x} - b(x)
\]

where \( b(x) \) denotes the residue class to which \( \frac{\pi^v(x)}{x} \) in \( k \). This gives rise to the continued fraction expansion of \( x \in \mathcal{M} \) in the form

\[
x = \frac{\pi^{a_1}}{b_1 + \frac{\pi^{a_2}}{b_2 + \frac{\pi^{a_3}}{b_3 + \ldots}}}
\]

where \( b_n \in k^\times \), \( a_n \in \mathbb{N} \) for \( n = 1, 2, \ldots \).
The start of continued fractions on a non-archimedean field

The rational approximants to $x \in \mathcal{M}$ arise in a manner similar to that in the case of the real numbers as follows. We suppose $A_0 = b_0$, $B_0 = 1$, $A_1 = b_0b_1 + \pi^{a_1}$, $B_1 = b_1$. Then set

$$A_n = \pi^{a_n}A_{n-2} + b_nA_{n-1} \text{ and } B_n = \pi^{a_n}B_{n-2} + b_nB_{n-1} \quad (5)$$

for $n \geq 2$. A simple inductive argument gives for $n = 1, 2, \ldots$

$$A_{n-1}B_n - A_nB_{n-1} = (-1)^n\pi^{a_1+\ldots+a_n}. \quad (6)$$
Dynamics of the Schneider’s map on a non-archimedean field

The map $T_v : \mathcal{M} \rightarrow \mathcal{M}$ preserves Haar measure on $\mathcal{M}$. We also have the following.

**Theorem**

Let $\mathcal{B}$ denote the Haar $\sigma$-algebra restricted to $\mathcal{M}$ and let $\mu$ denote Haar measure on $\mathcal{M}$. Then the measure-preserving transformation $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ has measure-theoretic entropy $\frac{|k|}{|k^x|} \log(|k|)$.

**Theorem**

Suppose $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ is as in our first theorem. Then the dynamical system $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ has a natural extension that is Bernoulli.

This tells us the isomorphism class of the dynamical system $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ is determined by its residue class field irrespective of the characteristic. This means for different $p$ each Schneider map on the $p$-adics non-isomorphic.
Theorem (Nair, Jaššová)

For any $L^p$-good universal sequence $(k_n)_{n \geq 1}$ where $(\{k_n \gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo one for each irrational number $\gamma$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_{k_n} = \frac{|k|}{|k \times|},$$

and

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} b_{k_n} = \frac{|k|}{2},$$

almost everywhere with respect to Haar measure on $\mathcal{M}$. 
Theorem (Nair, Jaššová)

For any $L^p$-good universal sequence $(k_n)_{n \geq 1}$ where $(\{k_n \gamma\})_{n=1}^{\infty}$ is uniformly distributed modulo one for each irrational number $\gamma$ we have

\[
\lim_{N \to \infty} \frac{1}{N} \# \{ 1 \leq n \leq N : a_{k_n} = i \} = \frac{|k^\times|}{|k|i};
\]

\[
\lim_{N \to \infty} \frac{1}{N} \# \{ 1 \leq n \leq N : a_{k_n} \geq i \} = \frac{1}{|k|i-1};
\]

\[
\lim_{N \to \infty} \frac{1}{N} \# \{ 1 \leq n \leq N : i \leq a_{k_n} < j \} = \frac{1}{|k|i-1} \left( 1 - \frac{1}{|k|j} \right);
\]

almost everywhere with respect to Haar measure on $\mathcal{M}$.
Thank you for your attention.