Essentially non-normal numbers for Cantor series expansions

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Definition

x is a normal in base s if for all k and blocks B of length k, one has

$$\lim_{n\to\infty} \frac{N_n^s(x,B)}{n} = \frac{1}{s^k}.$$

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x is simply normal in base s if it holds for $k = 1$.

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$\dim_H$ and Baire category of $[0,1] \setminus N_s$?

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L_s is a set of essentially non-normal numbers x if for any individual digit $i \in \{0, 1, \ldots, s-1\}$

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2017 Albeverio, Kondratiev, N., Torbin — $\dim_H$ of the set of essentially non-normal numbers is 1 for Q_∞-expansion (generalization of Lüroth expansion).

Cantor series expansion

If $Q \in \mathbb{N}_{\geq 2}^{\mathbb{N}}$, then we say that Q is a *basic sequence*.

Given a basic sequence $Q = (q_n)_{n=1}^{\infty}$, the *Q-Cantor series expansion* of a real number $x \in [0, 1]$ is the (unique) expansion of the form

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n}$$

where E_n is in $\{0, 1, \ldots, q_n - 1\}$ for $n \geq 1$ with $E_n \neq q_n - 1$ infinitely often. We abbreviate it with the notation $x = E_1 E_2 E_3 \ldots$ w.r.t. Q.

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For a basic sequence $Q=(q_n)$, a block $B=(b_1,b_2,\cdots,b_\ell)$, and a natural number j, define

$$I_j(B,Q)=\begin{cases}1 & \text{if } b_1<q_j, b_2<q_{j+1},\cdots,b_\ell<q_{j+\ell-1}\\ 0 & \text{otherwise}\end{cases}$$

and let

$$Q_n(B)=\sum_{j=1}^{n}\frac{I_j(B,Q)}{q_jq_{j+1}\cdots q_{j+\ell-1}}.$$

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A real number x is *Q-normal* if for all blocks B such that $\lim_{n\to\infty} Q_n(B)=\infty$

$$\lim_{n\to\infty}\frac{N_n^Q(B,x)}{Q_n(B)}=1.$$

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A real number x is Q-essentially non-normal if for all blocks B such $\lim_{n\to\infty} Q_n(B) = \infty$ the limit

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n(B)}$$

does not exist.

Let $L(Q)$ is a set of Q-essentially non-normal numbers.

Let X be a subshift of $\mathbb{N}_{\geq 2}^{\mathbb{N}}$, measure μ is fully supported in X, let basic sequence Q be a generic point for the dynamical system (X, T, μ).

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Theorem

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$L(Q)$ is the set of second Baire category.

Normal numbers along arithmetic progression

Let $m \in \mathbb{N}$ and $0 \leq r \leq m-1$.

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$$x = \sum_{i=1}^{\infty} \frac{\alpha_i(x)}{s^i}$$

is normal in base s, then

$$x_{m,r} = \sum_{t=0}^{\infty} \frac{\alpha_{mt+r}(x)}{s^{t+1}}$$

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For Cantor series expansion it does not true (Airey, Mance 2017)

Essentially non-normal numbers along arithmetic progression

For $m \in \mathbb{N}$ and $0 \leq r \leq m-1$ we define the basic sequence $\Lambda_{m,r}(Q) := (q_{mt+r})_{t=0}^{\infty}$.

If $x = E_1 E_2 \cdots$ w.r.t. Q, then let

$$\Upsilon_{Q,m,r}(x) := E_r E_{m+r} E_{2m+r} \cdots \text{ w.r.t. } \Lambda_{m,r}(Q).$$

Let

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Essentially non-normal numbers along arithmetic progression

$M = (m_t)_t$ is an increasing sequence of positive integers.
Given a sequence M, we define the basic sequence
$\Lambda_M(Q) := (q_{m_t})_{t=1}^{\infty}$.
If $x = E_1 E_2 \cdots$ w.r.t. Q, then let

$$\Upsilon_{Q,M}(x) := E_{m_1} E_{m_2} E_{m_3} \cdots \text{ w.r.t. } \Lambda_M(Q).$$

For $m \in \mathbb{N}$ and $0 \leq r \leq m-1$ let $A_{m,r} := (mt+r)_{t=0}^{\infty}$.

Let

$$L_{m,r}(Q) = \{\Upsilon_{Q,A_{m,r}}(x), x \in L_Q\}.$$

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$$\dim_H(L_{m,r}(Q)) = 1.$$

Theorem

Set $L_{m,r}(Q)$ is of second Baire category.

Idea of proof

We construct subset of $L(Q)$

$$
\begin{aligned}
L_s(Q) \quad = \quad & \big\{x:\ x\in(0,1),\\
& x=\underbrace{\alpha_{1,1}\alpha_{1,2}\ldots\alpha_{1,4s}\gamma_{1,1}\gamma_{1,2}01}_{\text{first group}}\\
& \underbrace{\alpha_{2,1}\alpha_{2,2}\ldots\alpha_{2,8s}\gamma_{2,1}\gamma_{2,2}\gamma_{2,3}\gamma_{2,4}0011}_{\text{second group}}\ldots\\
& \underbrace{\alpha_{k,1}\alpha_{k,2}\ldots\alpha_{k,2^{k+1}s}\gamma_{k,1}\ldots\gamma_{k,2^k}\overbrace{0\ldots0}^{2^{k-1}}\overbrace{1\ldots1}^{2^{k-1}}}_{k\text{-th group}}\ldots,\\
& \text{where } \alpha_{k,j}\in\{0,1,\ldots,q_{k,j}-1\},\forall k\in\mathbb{N}\big\}.
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Let $y=\gamma_{1,1}\gamma_{1,2}\gamma_{2,1}\gamma_{2,2}\gamma_{2,3}\gamma_{2,4}\ldots\gamma_{k,1}\ldots\gamma_{k,2^k}\ldots$ is normal number for the basic sequence $P_s=(Q_{1,s}Q_{2,s}\ldots)$, where $Q_{i,s}$ is a part of basic sequence Q in which positions digits of y are standing.

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$$\dim_H(L_s(Q)) \to 1, \; s \to \infty.$$

For proof $\dim_H(L_{m,r}(Q)) = 1$ we use

Theorem (V. Bergelson, J. Vandehey, paper under preparation)

Let (X, T, μ) is weak mixing.
If $x \in X$ is a normal number with symbolic expansion $[a_1, a_2, a_3, \ldots]$ and $y = [a_m, a_{m+r}, a_{m+2r}, ...]$, then the frequency of any block B in y exists.

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- Basic sequence Q is a shifted by 1 continued fraction normal number. For example, Adler–Keane–Smorodinsky number: concatenation of continued fractions of the rationals

$$\frac{1}{2},\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{2}{4},\frac{3}{4},\ldots$$

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- Q is a shifted by 2 Lüroth normal number or β-normal numbers, $\beta = \frac{1+\sqrt{5}}{2}$ (examples in Madritsch, Mance 2016)