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Table of contents

Invited talks

On spectra of numbers

Edita Pelantová
Czech Technical University in Prague

For a given base $\beta \in \mathbb{C}$ of modulus > 1 and a finite alphabet $\mathcal{A} \subset \mathbb{C}$, we consider the set

$$X^{\mathcal{A}}(\beta) = \Big\{ \sum_{k=0}^{n} a_k \beta^k \ : \ n \in \mathbb{N}, a_0, a_1, \ldots, a_n \in \mathcal{A} \Big\},$$

called $(\beta, \mathcal{A})$-spectrum.

We begin with the classical case where the base β is real and the alphabet $\mathcal{A}$ is subset of $\mathbb{Z}$. We review recent results on topological properties of the spectra. If β is a Pisot number, we discuss also geometrical properties of spectra and properties of symbolic sequences associated to $X^{\mathcal{A}}(\beta)$. Then we concentrate on $(\beta, \mathcal{A})$-spectra where β or $\mathcal{A}$ are complex. We motivate the study of complex spectra by questions comming from modeling of quasicrystals and from on-line algorithms for arithmetics in the complex field. We provide a list of results for complex spectra. It is modest and wide open to further research.

Normality, Computability and Discrepancy

Robert F. Tichy
Graz University of Technology

We give a survey on recent developments in the theory of normal numbers. In particular algorithms for the construction of absolutely normal numbers are established and analyzed. This includes normality with respect to classical radix system as well as with respect to Pisot numeration systems and continued fraction expansions. It seems that there is a tradeoff between the computational complexity of the algorithms and the speed of convergence to normality (measured by the discrepancy of the corresponding lacunary sequences). In paricular, constructions of Sierpinski, Turing, Schmidt and Levin are investigated in detail. Furthermore, various probabilistic limit laws for discrepancies of related sequences are established. This explains the statistical nature of normality and lacunarity.

Establishing and maintaining databases of self-affine tiles

Christoph Bandt (Greifswald, Germany), Dmitry Mekhontsev (Novosibirsk, Russia)

Self-affine graph-directed constructions (graph-IFS for short) include Rauzy-type tiles obtained from substitutions as well as Penrose-type tiles obtained from cut-and-project schemes. A projection approach based on mappings was implemented on computer by the second author to find large lists of new examples. Sometimes, especially in the fractal case, such lists become so extensive that careful visual inspection is impossible. Thus the computer must eliminate equivalent datasets and determine properties of the examples which can be used to select the most interesting specimen. We describe algorithms for the search as well as for the management of the database. The package is available at `https://ifstile.com`.

Our setup is a graph-IFS given by an expanding integer matrix M and integer translations in high-dimensional space. Tiles, or fractal attractors, are studied in a two-dimensional invariant subspace of M. Moreover, there can be a discrete symmetry group S which commutes with M, the simplest symmetry being $s^-(x) = -x$. Our figure shows a few modifications which are obtained from the Rauzy substitution matrix M with appropriate choice of translations and adding s^- to some of the contraction maps.

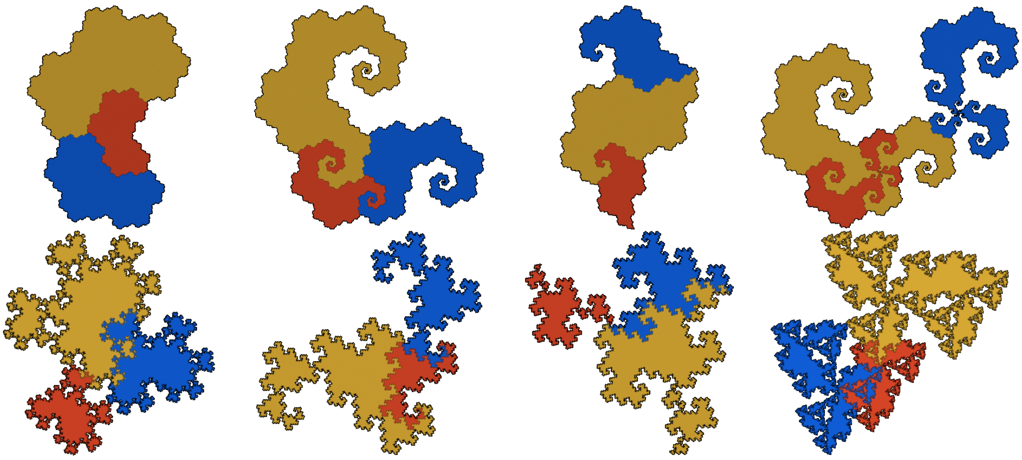

Figure 1: Simple modifications of the Rauzy tile.

The search for new examples is performed by a random walk on the parameter space of integer translations and discrete symmetries, while the given M and graph structure of the IFS remain fixed. The main algorithmic ingredient is a check for the open set condition of the graph-IFS which corresponds to the Arnoux-Ito coincidence property in the substitution approach. In contrast to the algorithms described in the monograph of Siegel and Thuswaldner (2010), we use only one automaton which we call the neighbor graph of the IFS. It describes in a canonical way the dynamical boundary of the given graph-IFS as a new graph-IFS.

To eliminate equivalent datasets from our list, we calculate various properties for each new dataset. Parameters of the neighbor graph, like number of vertices and edges, can be taken as invariants. The boundary dimension(s) and certain moments of the equidistribution on the tiles form other invariants. Topological properties can be calculated from the neighbor graph. The list of data sets can be sorted with respect to any property, and examples with desired properties can be selected.

In the talk, mathematical background will be given, and experiments with Rauzy-type fractals will be demonstrated.

Anne Bertrand Mathis
Université de Poitiers
anne.bertrand@math.univ-poitiers.fr

On the βexpansion of integers of $Z[\beta]$.Connection with the selfsimilar tilings

Let β be an unimodular Pisot number of degree d and let $\mathbb{Z}[\beta]$ be the set $\left\{u_1\beta^{d-1}+...+u_d; u_1,...,u_d\in\mathbb{Z}\right\}$. The βexpansions of the numbers of $\mathbb{Z}[\beta]$ are finite or ultimately periodic.

It is wellknown that if the β expansion of 1 is finite:

1) Every number of $\mathbb{Z}[\beta]$ admits a finite $\beta-$expansion (we say that β verifies the property F).

2) 0 is an interior point (in $\mathbb{R}^{d-1}$) of the fundamental tile of the Rauzy's selfsimilar tiling.

We prove that if the β expansion of 1 is $a_1...a_hb_1...b_kb_1...b_k...$ where $h+k=d$ (i.e. β do not has complementary values) then

1) Every numbers of $\mathbb{Z}[\beta]$ admits either a finite expansion or an expansion ending by $b_1...b_kb_1...b_k...$ (we says that β verifies the property $F^\star$).

2) 0 is an interior point (in $\mathbb{R}^{d-1}$) of the union of two tiles disjoints in measure: the fundamental tile and another tile related to $b_1...b_k$.

Looking at expansions in negative base $-\beta$ we say that β verifies the property F^- if all numbers of $\mathbb{Z}[\beta]$ admits a finite $-\beta$ expansion. We say that β verifies the property $F^{-\star}$ if there exists a periodic sequence $c_1...c_lc_1...c_l...$ such that all numbers of $\mathbb{Z}[\beta]$ admits either a finite $-\beta$ expansion or an expansion ending by $c_1...c_lc_1...c_l...$.

Now consider the periodic -β tiling and the selfsimilar $-\beta$ tiling of $\mathbb{R}^{d-1}$, who are simultaneously simple or multiple. We prove that

1) If β verifies the property F^- then the autosimilar $-\beta$ tiling is simple and 0 is an interior point of the fundamental tile of $\mathbb{R}^{d-1}$.

2) If β verifies the property $F^{-\star}$then the autosimilar $-\beta$ tiling is simple and 0 is an interior point of the union of two tiles of $\mathbb{R}^{d-1}$: the fundamental tile and another tile related to $c_1...c_lc_1...c_l....$.

Examples: the $-\beta$ tilings associated with quadratic unimodular numbers or with Tribonacci number are simple and it is also the case for those of Pisot number for wich the property F^- has been proved.

INTEGER MULTIPLIERS OF REAL POLYNOMIALS WITHOUT NONNEGATIVE ROOTS

HORST BRUNOTTE

For a polynomial f with real coefficients and positive leading coefficient the quantities

$$\delta(f) = \inf\{\deg(g) \ : \ g \in \mathbb{R}[X], \ gf \in \mathbb{R}_{>0}[X]\}$$

and

$$\delta_0(f) = \inf\{\deg(g) \ : \ g \in \mathbb{R}[X] \setminus \{0\}, \ gf \in \mathbb{R}_{\geq 0}[X]\}$$

have been introduced by J.-P. Borel [1]. It was shown by E. Meissner [4] and A. Durand (see [1, Théorème 2]) that $\delta(f)$ is finite if and only if f does not have a real nonnegative root; furthermore, $\delta_0(f)$ is finite if and only if f does not have a positive root. Moreover, given a polynomial $f \in \mathbb{R}[X]$ with positive leading coeffient, but without a real nonnegative root, a monic polynomial $t \in \mathbb{R}[X]$ with the properties

$$tf \in \mathbb{R}_{\geq 0}[X] \qquad \text{and} \qquad \deg(g) = \delta_0(f)$$

can effectively be computed; such a polynomial t is sometimes called a δ_0-multiplier of f. The analogous statement holds for f with positive leading coeffient and without a real positive root. We refer the reader to [3, 2] where also some more historical comments on theses quantities and their relations to algebraic number theory and distribution theory are given.

Having in mind these results, we here consider the set

$$\mathcal{F} = \{f \in \mathbb{R}[X] \ : \ f \text{ monic and } f \text{ does not have a root in } [0, \infty)\},$$

and for $f \in \mathcal{F}$ we ask for the quantities

$$\varphi(f) = \inf\{\deg(t) \ : \ t \in \mathbb{Z}[X], \ t \text{ monic and } tf \in \mathbb{R}_{>0}[X]\}$$

and

$$\varphi_0(f) = \inf\{\deg(t) \ : \ t \in \mathbb{Z}[X], \ t \text{ monic and } tf \in \mathbb{R}_{\geq 0}[X]\}.$$

Trivially, we have

$$\delta_0(f) \leq \varphi_0(f) \qquad \text{and} \qquad \delta(f) \leq \varphi(f).$$

We show that monic polynomials $s, t \in \mathbb{Z}[X]$ with the properties

$$tf \in \mathbb{R}_{\geq 0}[X] \qquad \text{and} \qquad sf \in \mathbb{R}_{>0}[X]$$

can effectively be computed thereby giving upper bounds for the constants $\varphi_0(f)$ and $\varphi(f)$.

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Haus-Endt-Strasse 88, D-40593 Düsseldorf, GERMANY
E-mail address: brunoth@web.de

On the magic of some families of fractal dendrites

Ligia L. Cristea
Institute for Mathematics and Scientific Computing,
University of Graz, Austria
(FWF-project P27050-N26)

Abstract

An $n \times n$ *pattern* is obtained by dividing the unit square into $n \times n$ congruent smaller sub-squares and colouring some of them in black (which means that they will be cut out), and the rest in white.

Sierpiński carpets are planar fractals that originate from the "classical" Sierpiński carpet. They are constructed in the following way: one starts with the unit square, divides it into $n \times n$ congruent smaller sub-squares and cuts out m of them, corresponding to a given $n \times n$ pattern (also called the generator of the Sierpiński carpet). This construction step is then repeated with all the remaining sub-squares ad infinitum. The resulting object is a self-similar fractal of Hausdorff and box-counting dimension $\log(n^2 - m)/\log(n)$, called a *Sierpiński carpet*.

By using special patterns, which we called "labyrinth patterns", we create and study a special class of carpets, called labyrinth fractals [1]. Labyrinth fractals are dendrites, i.e., connected and locally connected compact hausdorff spaces that contain no simple curves. Under certain conditions on the patterns one obtains objects with some "magic" properties. First, we study the self-similar case. Already in this "simplest" case one needs results from several areas of mathematics (topology, combinatorics, linear algebra, curves theory, graph theory) to establish the main results. An important role is played here by the path matrix of a pattern or a labyrinth set. This is in fact the matrix of a subsitution and is, for an important class of labyrinth patterns, primitive [2].

As a next step, we introduce and study mixed labyrinth fractals [3], which are not self-similar. It is interesting to see here which properties are inherited from the self-similar case, and which are not.

The results obtained show how by an appropriate choice of the labyrinth patterns, one can obtain . . . almost anything [4].

In very recent research [5] we study an even more general class, called supermixed labyrinth fractals, and solve a conjecture on mixed labyrinth fractals. Every time we pass to a more general class, it was necessary to introduce new objects and tools and use new techniques for our proofs.

Wild labyrinth fractals are a further generalisation ...

It is worth mentioning that some of our results on labyrinth fractals have already been used by physicists in their research and construction of prototypes. Moreover, we are aware that these objects are suitable as future models for certain crystals, as other recent research in physics shows.

The results stem from joint work with Bertran Steinsky and Gunther Leobacher.

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MULTIPLICATIVE DEPENDENCE OF SHIFTED ALGEBRAIC NUMBERS

ARTŪRAS DUBICKAS

Given $n \geq 2$ non-zero complex numbers $z_1, z_2, \ldots, z_n$, we say that they are *multiplicatively dependent* if there exists a non-zero integer vector $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$ for which

$$z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n} = 1.$$

Otherwise (if there is no such non-zero integer vector $(k_1, k_2, \ldots, k_n)$), we say that the numbers $z_1, z_2, \ldots, z_n$ are *multiplicatively independent.*

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be n pairwise distinct algebraic numbers. In [1], we show that the numbers

$$\alpha_1 + t, \alpha_2 + t, \ldots, \alpha_n + t$$

are multiplicatively independent for all sufficiently large positive integers t. More generally,

Theorem 1. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be pairwise distinct algebraic numbers, and let $d = [\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n) : \mathbb{Q}]$. Then, there is a positive constant $C = C(n, \alpha_1, \alpha_2, \ldots, \alpha_n)$ such that, for any algebraic integer t of degree at most $\overline{|t|}^{1/(nd+1)}$, where $\overline{|t|} \geq C$ (here $\overline{|t|}$ is the largest modulus of the conjugates of t over $\mathbb{Q}$), the algebraic numbers $\alpha_1 + t, \alpha_2 + t, \ldots, \alpha_n + t$ are multiplicatively independent.*

A more general approach, with rational functions instead of translations $\alpha_i + t$, has been considered in [2].

In [1], for a pair (a, b) of distinct integers $a < b$, we also study how many pairs $(a + t, b + t)$ are multiplicatively dependent when t runs through the set integers $\mathbb{Z}$. Assuming the ABC conjecture, we show that there exists a constant C_0 such that for any pair $(a, b) \in \mathbb{Z}^2$, $a < b$, there are at most C_0 values of $t \in \mathbb{Z}$ for which the pairs $(a+t, b+t)$ are multiplicatively dependent. (Without ABC, the bound on the number of such $t \in \mathbb{Z}$ depends on the number of distinct prime divisors of $b-a$.)

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Key words and phrases. Multiplicative dependence, multiplicative independence, Pillai's equation, ABC conjecture.

For a pair $(a,b) \in \mathbb{Z}^2$ with difference $b-a=30$, we show that there are 13 values of $t \in \mathbb{Z}$ for which the pairs $(a+t, b+t)$ are multiplicatively dependent, namely,

$$(-15,15), (-1,29), (-29,1), (1,31), (-31,-1), (-5,25), (-25,5),$$
$$(-3,27), (-27,3), (2,32), (-32,-2), (6,36), (-36,-6).$$

We conjecture that 13 is the largest number of such translations for any such pair $(a,b) \in \mathbb{Z}^2$, $a<b$, and that the number 13 is attained only when the difference $b-a$ equals 30. This was proved for all pairs (a,b) with difference at most 10^{10}.

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Institute of Mathematics, Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania

E-mail address: `arturas.dubickas@mif.vu.lt`

Representations of palindromes in the Fibonacci word

Anna E. Frid

Abstract

We continue our study of palindromes in Sturmian words in terms of Ostrowski representations of their ends. In the previous preprint, we used the link between them to prove that every Sturmian word with an unbounded directive sequence contains a factor which cannot be decomposed to a concatenation of a given number of palindromes. However, for the case of bounded directive sequence, the only existing proof of this property is still the non-constructive one dated 2013. So, in this study we start considering in detail palindromes in the Fibonacci word in terms of the (Lekkerkerker-)Zeckendorf representations of their ends. In particular, we prove an upper bound for the number of palindromes necessary to construct the Fibonacci prefix of length n and conjecture which prefixes are the shortest with a given number of palindromes.

The *palindromic length* of a finite word u is the minimal number Q of palindromes $P_1, \dots, P_Q$ such that $u = P_1 \cdots P_Q$.

Conjecture 1. *In every infinite word which is not ultimately periodic, the palindromic length of factors is unbounded.*

The conjecture was stated in 2013 by Puzynina, Zamboni and the author [5] and was proved in the same paper for the case when the infinite word is k-power-free for some k. To prove the conjecture at least for Sturmian words which are not k-power-free, in the previous paper [4] the author managed to express an occurrence of a palindrome in a characteristic Sturmian word in terms of the Ostrowski representations of its ends. The first result of this abstract is a new, simplified and updated, version of the respective theorem, which we will formulate after introducing the notation.

As usual, we use in this paper the classical construction of characteristic Sturmian words related to a *directive sequence* $(d_0, d_1, \dots, d_n, \dots)$, where

$d_i \geq 1$. Given a directive sequence, the *standard sequence* (s_n) of words on the binary alphabet $\{a, b\}$ is defined as follows:

$$s_{-1} = b, s_0 = a, s_{n+1} = s_n^{d_n} s_{n-1} \text{ for all } n \geq 0. \tag{1}$$

The word s_n is called also the *standard word of order n*. The infinite word $w = \lim_{n\to\infty} s_n$ is called a *characteristic Sturmian word* associated with a sequence (d_i).

Note that to get all possible Sturmian words, we also need to allow $d_0 = 0$, but due to the symmetry between a and b, we can restrict ourselves to the case of $d_1 > 0$.

Denote the length of s_n by q_n; then, clearly,

$$q_{-1} = q_0 = 1, q_{n+1} = d_n q_n + q_{n-1} \text{ for all } n \geq 0.$$

In the *Ostrowski numeration system* [7] associated with the sequence (d_i), a non-negative integer $N < q_{j+1}$ is represented as

$$N = \sum_{0\leq i\leq n} k_i q_i, \tag{2}$$

where $0 \leq k_i \leq d_i$ for $i \geq 0$, and for $i \geq 1$, if $k_i = d_i$, then $k_{i-1} = 0$. Such a representation of N is unique up to leading zeros; we use the notation $N = \overline{k_n \cdots k_1 k_0}[o]$. Everywhere in the text, we will not distinguish representations which differ only by leading zeros.

In many cases, including ours, it is reasonable to consider more general *legal* decompositions $N = \sum_{0\leq i\leq n} k_i q_i$, where $0 \leq k_i \leq d_i$ for $i \geq 0$ (but the second restriction from the definition of the Ostrowski representation is not imposed). A number can admit several legal representations, including the Ostrowski one. A legal representation of N is denoted by $N = \overline{k_n \cdots k_1 k_0}$ (without $[o]$ at the end, reserved for the Ostrowski version).

Proposition 1. *[4] For all $k_0, \dots, k_n$ such that $k_i \leq d_i$, the word $s_n^{k_n} s_{n-1}^{k_{n-1}} \cdots s_0^{k_0}$ is a prefix of w.*

The following statement is an updated and simplified version of Theorem 2 from [4].

Theorem 1. *Let w be a characteristic Sturmian word corresponding to the directive sequence (d_n), and $w(i..j] = w[i+1]\dots w[j]$ be a palindrome. Denote the Ostrowski representation of i as $i = \overline{x_n \cdots x_m \cdots x_0}[o]$; note that it* *may start with several leading zeros. Then there exist a legal representation of j given by*

$$j = \overline{x_n \cdots x_{m+1} y_m \cdot (d_{m-1} - x_{m-1}) \cdots (d_0 - x_0)},$$

where $0 \leq m \leq n$ and $x_m < y_m \leq d_m$.

As a classical example, consider the directive sequence $(1, 1, 1, \ldots)$. It corresponds to the famous Fibonacci word defined by its prefixes $s_0 = a$, $s_1 = s_0 s_{-1} = ab$, $s_2 = s_1 s_0 = aba$, $s_3 = s_2 s_1 = abaab$, etc.:

$$w = abaababaabaababaababa \cdots .$$

The lengths $q_i = |s_i|$ are Fibonacci numbers, and the respective numeration system is the Fibonacci, or Zeckendorf, one [6, 8]: it corresponds to the greedy decomposition of a number N to a sum of Fibonacci numbers F_n, starting from $F_0 = 1$, $F_1 = 2$.

Consider the palindrome $w(12..13] = w[13] = b$ in the Fibonacci word. The Ostrowski representation of 12 is $12 = \overline{10101}[o]$. Taking $m = 1$ and applying the procedure from the previous theorem, we increase the second-to last symbol of the representation from 0 to 1 (this is the only possible option in the Fibonacci case) and invert the last symbol. So, get the representation $\overline{10110} = 13 = \overline{100000}[o]$.

For the palindrome $w(5, 14] = abaabaaba$, we take the Ostrowski representation of 5 with a leading zero: $5 = \overline{1000}[o] = \overline{01000}[o]$. Now we add 1 to the first zero of this long representation and invert everything which follows. It gives the representation $\overline{10111} = \overline{100001}[o] = 14$.

The following statement is partially obtained by a computer case study.

Proposition 2. *The smallest lengths of prefixes of the Fibonacci word not decomposable as a concatenation of K palindromes, $K = 1, \ldots, 6$, are, in the Ostrowski representation, respectively $\overline{10} = 2$, $\overline{10001} = 9$, $\overline{100001010}$, $\overline{100100010001}$, $\overline{100010001000010}$, $\overline{100100100100100101}$.*

Note that the same result had been already known to Bucci and Richomme in 2016 [2]. However, more data we got with this Ostrowski technique allows to prove by induction at least one infinite statement on the palindromic length of prefixes of the Fibonacci word.

Proposition 3. *For every $k \geq 3$, all prefixes of the Fibonacci word of length smaller than $\overline{(100)^{2k-1}101}$ can be decomposed as a concatenation of at most $2k$ palindromes.*

The result would be complete if we manage to prove the following conjecture.

Conjecture 2. *For every $k \geq 1$, the prefix of the Fibonacci word of length $\overline{(100)^{2k-1}101}$ cannot be decomposed as a concatenation of at most $2k$ palindromes.*

We proved that $2k+1$ palindromes are enough for this word, so, it remains just to prove that this is the minimal possible value.

The analogous conjecture for the odd number $2k+1$ of palindromes will probably mention the critical length $\overline{(100)^{2k}010}$ starting from some k.

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RANDOM HOMOGENEOUS BETA-EXPANSIONS AND SELF-SIMILAR MEASURES

KEVIN G. HARE

This talk will discuss joint work with Kathryn Hare and Sascha Troscheit. Let $\beta \in (1, 2)$ and consider the expansion

$$x = \sum_{j=1}^{\infty} a_j \beta^{-j}$$

where $a_j \in \{0, 1\}$. Then $a_1 a_2 a_3 \cdots$ is a *beta-expansion* for x. Beta-expansions, and their associated self-similar measures are well studied within the literature.

In this talk we consider a variation of the classical beta-expansion, one that is *random homogeneous*. As a simple example, let S_1 and S_2 be finite sets. Consider the set K of all x such that

$$x = \sum_{j=1}^{\infty} a_j^{(i)} \beta^{-j}$$

where $a_j^{(i)} \in S_{b_i}$ where the b_i are choosen randomly from $\{1, 2\}$. We see that if $a_j = 1$ for all j then K would be the set of allowable beta expansions with digit set S_1. Similarly if $a_j = 2$ for all j, then K would be based on digit set S_2. As we are choosing a_j randomly, this is a hybrid between the two types of beta-expansions. Then we investigate various almost sure properties of K.

TOTALLY POSITIVE QUADRATIC INTEGERS AND NUMERATION

TOMÁŠ HEJDA AND VÍTĚZSLAV KALA

Abstract. Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field. We obtain a presentation of the additive semigroup $\mathcal{O}_K^+(+)$ of totally positive integers in K; its generators (indecomposable integers) and relations can be nicely described in terms of the periodic continued fraction for $\sqrt{D}$. We also characterize all uniquely decomposable integers in K. In these results, we make use of an integer numeration system we construct on $\mathcal{O}_K$.

1. Introduction

The additive semigroup of totally positive integers $\mathcal{O}_K^+$ in a totally real number field K has long played a fundamental role in algebraic number theory, even though more attention has perhaps been paid to the multiplicative structure of the ring $\mathcal{O}_K$, for example, to its units and unique factorization into primes. Most prominent purely additive objects are the *indecomposable elements*, i.e., totally positive integers $\alpha \in \mathcal{O}_K^+$ that cannot be decomposed into a sum $\alpha = \beta + \gamma$ of totally positive integers $\beta, \gamma \in \mathcal{O}_K^+$.

In the real quadratic case $K = \mathbb{Q}(\sqrt{D})$, indecomposables can be nicely characterized in terms of continued fraction (semi-)convergents to $\sqrt{D}$. This stands in contrast to the situation of a general totally real field K, where it is much harder to describe indecomposables: Brunotte [Bru83] proved an upper bound on their norm in terms of the regulator, but otherwise their structure remains quite mysterious.

The goal of this work is to study the structure of the whole additive semigroup $\mathcal{O}_K^+(+)$. This is an interesting problem in itself, but it seems also necessary for certain applications (such as the recent progress in the study of universal quadratic forms and lattices over K by Kim, Blomer, Yatsyna, and the second author [BK15, BK17, Kal16, KY17, Kim00]).

In particular, as indecomposable elements are precisely the generators of $\mathcal{O}_K^+(+)$, we need to determine the relations between them. While the description of indecomposables in terms of the continued fraction is fairly straightforward, it is a priori not clear at all if the same will be the case for relations, as there could be some "random" or "accidental" ones. Perhaps surprisingly, it turns out that this is not the case and that the presentation of the semigroup $\mathcal{O}_K^+(+)$ (given in Theorem 1) is quite elegant. A key tool in the proof of the presentation is the fact that each totally positive integer can be uniquely written as a $\mathbb{Z}^+$-linear combination of two consecutive indecomposables (Proposition 2).

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We also manage to find a basis, alphabet and forbidden strings for a numeration system on $\mathcal{O}_K^+$ and show some of its properties (§ 4). This numeration system naturally generalizes integer numeration systems (see, e.g., [Ost22, Zec72, Fro92, GT91]) from $K = \mathbb{Q}$ to real quadratic fields; the price is that the base becomes a bi-infinite sequence, namely it is the indecomposables that will play the role of the basis.

One of course cannot hope to have an analogue of unique factorization in the additive setting, but nevertheless, some elements can be uniquely decomposed as a sum of indecomposables. In Theorem 6 we characterize all such *uniquely decomposable elements* and obtain again a very explicit result depending only on the continued fraction. This then yields directly Corollary 7 that $\mathcal{O}_K^+(+)$ (viewed as an abstract semigroup) completely determines D and the number field K.

2. Preliminaries

Throughout the work, we will use the following notation. We fix a squarefree integer $D \geq 2$ and consider the real quadratic field $K = \mathbb{Q}(\sqrt{D})$ and its ring of integers $\mathcal{O}_K$; we know that $\{1, \omega_D\}$ forms an integral basis of $\mathcal{O}_K$, where

$$\omega_D := \begin{cases} \sqrt{D} & \text{if } D \equiv 2, 3 \pmod 4, \\ \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \pmod 4. \end{cases}$$

By Δ we denote the discriminant of K, i.e., $\Delta = 4D$ if $D \equiv 2, 3 \pmod 4$ and $\Delta = D$ otherwise. The norm and trace from K to $\mathbb{Q}$ are denoted by N and Tr, respectively.

An algebraic integer $\alpha \in \mathcal{O}_K$ is *totally positive* iff $\alpha > 0$ and $\alpha' > 0$, where α' is the Galois conjugate of α, we write this fact as $\alpha \succ 0$; for $\alpha, \beta \in \mathcal{O}_K$ we denote by $\alpha \succ \beta$ the fact that $\alpha - \beta \succ 0$, and by $\mathcal{O}_K^+$ the set of all totally positive integers. We say that $\alpha \in \mathcal{O}_K^+$ is *indecomposable* iff it can not be written as a sum of two totally positive integers or equivalently iff there is no algebraic integer $\beta \in \mathcal{O}_K^+$ such that $\alpha \succ \beta$. We say that $\alpha \in \mathcal{O}_K^+$ is *uniquely decomposable* iff there is a unique way how to express it as a sum of indecomposable elements.

It will be slightly more convenient for us to work with a purely periodic continued fraction, and so let $\sigma_D = [\overline{u_0, u_1, \dots, u_{s-1}}]$ be the periodic continued fraction expansion of

$$\sigma_D := \omega_D + \lfloor -\omega_D' \rfloor = \begin{cases} \sqrt{D} + \lfloor \sqrt{D} \rfloor & \text{if } D \equiv 2, 3 \pmod 4, \\ \frac{1+\sqrt{D}}{2} + \left\lfloor \frac{-1+\sqrt{D}}{2} \right\rfloor & \text{if } D \equiv 1 \pmod 4 \end{cases}$$

(with positive integers u_i). We then have that $\omega_D = \left[\lceil u_0/2 \rceil, \overline{u_1, \dots, u_s}\right]$. It is well known that $u_1, u_2, \dots, u_{s-1}$ is a palindrome and that $u_0 = u_s$ is even if and only if $D \equiv 2, 3 \bmod 4$, hence $\lceil u_0/2 \rceil = (u_s + \mathrm{Tr}(\omega_D))/2$.

Denote the convergents to ω_D by $p_i/q_i := \left[\lceil u_0/2 \rceil, u_1, \dots, u_i\right]$ and recall that the sequences (p_i), (q_i) satisfy the recurrence

$$X_{i+2} = u_{i+2} X_{i+1} + X_i \quad \text{for} \quad i \geq -1 \tag{1}$$

with the initial condition $q_{-1} = 0$, $p_{-1} = q_0 = 1$, and $p_0 = \lceil u_0/2 \rceil$. Denote $\alpha_i := p_i - q_i \omega_D'$ and $\alpha_{i,r} = \alpha_i + r\alpha_{i+1}$. Then we have the following classical facts (see, e.g., [DS82, Per13]):

- The sequence (α_i) satisfies the recurrence (1).
- $\alpha_i \succ 0$ if and only if $i \geq -1$ is odd.

- The indecomposable elements in $\mathcal{O}_K^+$ are $\alpha_{i,r}$ with odd $i \ge -1$ and $0 \le r \le u_{i+2} - 1$, together with their conjugates.
- We have that $\alpha_{i,u_{i+2}} = \alpha_{i+2,0}$.
- The indecomposables $\alpha_{i,r}$ are increasing with increasing (i,r) (in the lexicographic sense).
- The indecomposables $\alpha'_{i,r}$ are decreasing with increasing (i,r).

We also denote $\varepsilon > 1$ the fundamental unit of $\mathcal{O}_K$; we have that $\varepsilon = \alpha_{s-1}$. Furthermore, we denote $\varepsilon^+ > 1$ the smallest totally positive unit > 1; we have that $\varepsilon^+ = \varepsilon$ if s is even and $\varepsilon^+ = \varepsilon^2 = \alpha_{2s-1}$ if s is odd. Furthermore, we denote $\gamma_0 = \omega_D$ and $\gamma_i = [u_i, u_{i+1}, u_{i+2}, \dots]$ for $i \ge 1$; we have that $u_i < \gamma_i = u_i + \frac{1}{\gamma_{i+1}} < u_i + 1$ for $i \ge 1$.

3. Presentation of the semigroup $\mathcal{O}_K^+(+)$

In this section we will prove the following theorem that gives a presentation of the semigroup $\mathcal{O}_K^+(+)$. We recall that $\langle S \mid \mathcal{R} \rangle$ is a presentation of a semigroup $G(+)$ iff G is generated by S and all (additive) relations between elements of S are generated by the relations in $\mathcal{R}$.

Theorem 1. *The additive semigroup $\mathcal{O}_K^+(+)$ is presented by*

$$\mathcal{O}_K^+ = \big\langle \mathcal{A} \cup \mathcal{A}' \cup \{1\} \bigm| \mathcal{R}, \mathcal{R}', \mathcal{R}_0 \big\rangle,$$

where $\mathcal{A} := \{\, \alpha_{i,r} : i \ge -1$ odd and $0 \le r \le u_{i+2} - 1 \,\} \setminus \{1\}$ are the indecomposable elements > 1, $\mathcal{A}' := \{\, y' : y \in \mathcal{A} \,\}$, and the relations are the following:

$$\begin{aligned} (2) \qquad & \mathcal{R}\colon \alpha_{i,r-1} - 2\alpha_{i,r} + \alpha_{i,r+1} = 0 \quad \text{for odd} \quad i \ge -1 \text{ and } 1 \le r \le u_{i+2} - 1, \\ & \qquad \alpha_{i-2,u_i-1} - (u_{i+1}+2)\alpha_{i,0} + \alpha_{i,1} = 0 \quad \text{for odd} \quad i \ge 1; \\ & \mathcal{R}'\colon \text{same relations as in } \mathcal{R} \text{ after applying the isomorphism } ('); \\ & \mathcal{R}_0\colon \alpha'_{-1,1} - (u_0+2)\cdot 1 + \alpha_{-1,1} = 0. \end{aligned}$$

For convenience, we introduce an alternative notation of the indecomposables. We define β_j, $j \in \mathbb{Z}$ by the condition that $\cdots < \beta_{-3} < \beta_{-2} < \beta_{-1} < \beta_0 = 1 < \beta_1 < \beta_2 < \beta_3 < \cdots$ is the increasing sequence of the indecomposables. Note that we have $\beta'_j = \beta_{-j}$ for all $j \in \mathbb{Z}$. Using this notation, we can rewrite the relations (2) in a unified way in terms of β_j and certain constants v_j as follows:

$$(3) \qquad \mathcal{R}, \mathcal{R}', \mathcal{R}_0\colon \beta_{j-1} - v_j\beta_j + \beta_{j+1} = 0 \quad \text{for} \quad j \in \mathbb{Z}.$$

Note that the given set of relations is minimal in the sense that none of them can be removed.

In order to prove the previous theorem, we provide a way of expressing totally positive integers as sums of indecomposables: A variant of this concerning sums of powers of units was used by Kim, Blomer, and the second author [BK17, Kim00] in the construction of universal quadratic forms.

Proposition 2. *Let $x \in \mathcal{O}_K^+$. Then there exist unique $j_0, e, f \in \mathbb{Z}$ with $e \ge 1$ and $f \ge 0$ such that $x = e\beta_{j_0} + f\beta_{j_0+1}$;*

4. Numeration system on $\mathcal{O}_K^+$

The numeration system we are about to define resembles the Ostrowski numeration systems and is its generalization onto integer rings other than $\mathbb{Z}$.

Proposition 3. *The map $(l_j)_{j\in\mathbb{Z}} \mapsto \sum_{j\in\mathbb{Z}} l_j\beta_j$ is a bijection $\mathcal{L} \to \mathcal{O}_K^+$, where $\mathcal{L}$ is the set of sequences $(l_j)_{j\in\mathbb{Z}}$ satisfying the following:*

- *$l_j \in \mathbb{Z}$ and only finitely many l_j's are non-zero;*
- *$0 \le l_j \le v_j - 1$ for all $j \in \mathbb{Z}$;*
- *for all $j, J \in \mathbb{Z}$, $j \le J-1$ we have that $(l_j, l_{j+1}, \dots, l_J) \neq (v_j - 1, v_{j+1} - 2, v_{j+2} - 2, \dots, v_{J-2} - 2, v_{J-1} - 2, v_J - 1)$.*

We will call the sequence $(l_j)_{j\in\mathbb{Z}} \in \mathcal{L}$ the *greedy expansion* of x. It also satisfies the following:

Theorem 4. (1) *The greedy expansion of $x \in \mathcal{O}_K^+$ is obtained by the usual greedy algorithm, where the seqence $(\beta_j)_{j\in\mathbb{Z}}$ is taken as the base.*

(2) *Suppose $x \in \mathcal{O}_K^+$ is a finite sum $x = \sum_{j\in\mathbb{Z}} k_j\beta_j$ with integers $k_j \ge 0$. Then the greedy expansion of x is obtained by repeated applying of the relations* (3) *on this sum at any position with $k_j \ge v_j$ as long as such a position exists; this process is terminated.*

(3) *A sequence $(l_j)_{j\in\mathbb{Z}}$ is the greedy expansion of $x \in \mathcal{O}_K^+$ if and only if $(l_{-j})_{j\in\mathbb{Z}}$ is the greedy expansion of $x' \in \mathcal{O}_K^+$.*

5. Uniquely decomposable elements of $\mathcal{O}_K^+$

The greedy expansions allow us to describe all uniquely decomposable elements of $\mathcal{O}_K^+$:

Proposition 5. *A number $x \in \mathcal{O}_K^+$ is uniquely decomposable if and only if its greedy expansion $(l_j)_{j\in\mathbb{Z}}$ is of length one or two, i.e., if there exists j_0 such that $l_j = 0$ for all $j \neq j_0, j_0 + 1$.*

From this, we can list them explicitly:

Theorem 6. *All uniquely decomposable elements $x \in \mathcal{O}_K^+$ are the following:*

(a) *$\alpha_{i,r}$ with odd $i \ge -1$ and $0 \le r \le u_{i+2} - 1$;*

(b) *$e\alpha_{i,0}$ with odd $i \ge -1$ and with $2 \le e \le u_{i+1} + 1$*

(c) *$\alpha_{i,u_{i+2}-1} + f\alpha_{i+2,0}$ with odd $i \ge -1$ odd such that $u_{i+2} \ge 2$ and with $1 \le f \le u_{i+3}$;*

(d) *$e\alpha_{i,0} + \alpha_{i,1}$ with odd $i \ge -1$ such that $u_{i+2} \ge 2$ and with $1 \le e \le u_{i+1}$;*

(e) *$e\alpha_{i,0} + f\alpha_{i+2,0}$ with odd $i \ge -1$ such that $u_{i+2} = 1$ and with $1 \le e \le u_{i+1}+1$, $1 \le f \le u_{i+3} + 1$, $(e, f) \neq (u_{i+1} + 1, u_{i+3} + 1)$;*

(f) *Galois conjugates of all of the above.*

Corollary 7. *The additive semigroups $\mathcal{O}_K^+$, for real quadratic fields K, are pairwise not isomorphic.*

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Charles University, Faculty of Mathematics and Physics, Department of Algebra, Sokolovská 83, 18600 Praha 8, Czech Republic

University of Chemistry and Technology, Prague, Department of Mathematics, Studentská 6, 16000 Praha 6, Czech Republic

E-mail address: `tohecz@gmail.com`

Charles University, Faculty of Mathematics and Physics, Department of Algebra, Sokolovská 83, 18600 Praha 8, Czech Republic

University of Göttingen, Mathematisches Institut, Bunsenstr. 3–5, D-37073 Göttingen, Germany

E-mail address: `vita.kala@gmail.com`

Toolset for supporting the number system research

Péter Hudoba and Attila Kovács

Abstract The world of generalized number systems contains many challenging areas. In some cases the complexity of the arising problems is unknown, computer experiments are able to support the theoretical research. In this talk we introduce a new toolset that helps to analyze number systems in lattices. The toolset is able to

- Analyze the expansions;
- Decide the number system property;
- Classify and visualize the periodic points;
- Calculate correlations between system data, etc.

In this talk we present an introductory usage of the toolset.

1 Extended abstract

1.1 Introduction

Let Λ be a lattice in $\mathbb{R}^n$ and let $M : \Lambda \to \Lambda$ be a linear operator such that $\det(M) \neq 0$. Let furthermore $0 \in D \subseteq \Lambda$ be a finite subset. Lattices can be seen as finitely generated free Abelian groups. In this talk we consider number expansions in lattices.

Definition 1. The triple (Λ, M, D) is called a *number system* (GNS) if every element x of Λ has a unique, finite representation of the form

Péter Hudoba
Eötvös Loránd University, Budapest Hungary, e-mail: peter.hudoba@inf.elte.hu

Attila Kovács
Eötvös Loránd University, Budapest Hungary e-mail: attila.kovacs@inf.elte.hu

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$$x = \sum_{i=0}^{L} M^i d_i \ ,$$

where $d_i \in D$ and $L \in N$.

Here M is called the *base* and D is the *digit set*. It is easy to see that similarity preserves the number system property, i.e., if M_1 and M_2 are similar via the matrix Q then (Λ, M_1, D) is a number system if and only if $(Q\Lambda, M_2, QD)$ is a number system at the same time. If we change the basis in Λ a similar integer matrix can be obtained, hence, no loss of generality in assuming that M is integral acting on the lattice Z^n.

If two elements of Λ are in the same coset of the factor group $\Lambda/M\Lambda$ then they are said to be *congruent* modulo M. If (Λ, M, D) is a number system then

1. D must be a full residue system modulo M;
2. M must be expansive;
3. $\det(I_n - M) \neq \pm 1$ (*unit condition*) .

If a system fulfils the first two conditions then it is called a *radix system*.
Let $\varphi : \Lambda \to \Lambda$, $x \overset{\varphi}{\mapsto} M^{-1}(x-d)$ for the unique $d \in D$ satisfying $x \equiv d \pmod{M}$. Since M^{-1} is contractive and D is finite, there exists a norm $\|.\|$ on Λ and a constant C such that the orbit of every $x \in \Lambda$ eventually enters the finite set $S = \{x \in \Lambda \mid \|x\| < C\}$ for the repeated application of φ. This means that the sequence $x, \varphi(x), \varphi^2(x), \dots$ is eventually periodic for all $x \in \Lambda$. Clearly, (Λ, M, D) is a number system iff for every $x \in \Lambda$ the orbit of x eventually reaches 0. A point p is called *periodic* if $\varphi^k(p) = p$ for some $k > 0$. The orbit of a periodic point p is a *cycle*. The set of all periodic points is denoted by $\mathscr{P}$. The *signature* $(l_1, l_2, \dots, l_\omega)$ of a radix system is a finite sequence of non-negative integers in which the periodic structure $\mathscr{P}$ consists of $\#l_i$ cycles with period length i $(1 \leq i \leq \omega)$.

The following problem classes are in the mainstream of the research: for a given (Λ, M, D)

- the *decision problem* asks if the triple form a number system or not;
- the *classification problem* means finding all cycles (*witnesses*);
- the *parametrization problem* means finding parametrized families of number systems;
- the *construction problem* aims at constructing a digit set D to M for which (Λ, M, D) is a number system. In general, construct a digit set D to M such that (Λ, M, D) satisfies a given signature.

The algorithmic complexity of the decision and classification problems is still unknown.

1.2 The toolset

We implemented a python based system that contain the following:

- Base features of number systems (addition, multiplication);
- Procedures for the decision and the classification problems;
- Visualization for structure analysis, fractions, etc.
- Different optimization algorithms, etc.

Moreover, in order to support the research we implemented a serves-side application which is able to store various data on predefined systems. The database already contains more than 10000 candidates of the following type:

- Special bases (companions of expansive polynomials with constant terms ± 2,± 3,± 5, ± 7) with canonical, shifted canonical, symmetric digit sets.
- Product systems.

The data server allows to read the data from the server publicly via a JSON API. The registered users with API token can send new properties also to the database and store them in a flexible way. The server already stores plenty of properties, for example

- Eigenvalues, eigenvectors,
- Periodic points and orbits,
- Classification details.

In this talk we present an introductory usage of the toolset.

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Characterization of rational matrices that admit finite digit representations

J. Jankauskas and J. M. Thuswaldner

Abstract Let A be an $n \times n$ matrix with rational entries and let

$$\mathbb{Z}^n[A] := \bigcup_{k=1}^{\infty} \left(\mathbb{Z}^n + A\mathbb{Z}^n + \ldots + A^{k-1}\mathbb{Z}^n \right)$$

be the minimal A-invariant $\mathbb{Z}$-module containing the lattice $\mathbb{Z}^n$. If $\mathscr{D} \subset \mathbb{Z}^n[A]$ is a finite set we call the pair $(A, \mathscr{D})$ *a digit system*. We say that $(A, \mathscr{D})$ has *the finiteness property* if each $\mathbf{z} \in \mathbb{Z}^n[A]$ can be written in the form

$$\mathbf{z} = \mathbf{d}_0 + A\mathbf{d}_1 + \ldots + A^k \mathbf{d}_k,$$

with $k \in \mathbb{N}$ and *digits* $\mathbf{d}_j \in \mathscr{D}$ for $0 \le j \le k$. We prove that for a given matrix $A \in M_n(\mathbb{Q})$ there is a finite set $\mathscr{D} \subset \mathbb{Z}^n[A]$ such that $(A, \mathscr{D})$ has the finiteness property if and only if A has no eigenvalue of absolute value < 1. This result is the matrix analogue of *the height reducing property* of algebraic numbers. In proving this result we also characterize integer polynomials $P \in \mathbb{Z}[x]$ that admit digit systems having the finiteness property in the quotient ring $\mathbb{Z}[x]/(P)$.

J. Jankauskas
Mathematik und Statistik, Montanuniversität Leoben, Franz Josef Straße 18, A-8700 Leoben, Austria, e-mail: jonas.jankauskas@gmail.com

J. M. Thuswaldner
Mathematik und Statistik, Montanuniversität Leoben, Franz Josef Straße 18, A-8700 Leoben, Austria, e-mail: joerg.thuswaldner@unileoben.ac.at

UNIQUE EXPANSIONS ON FAT SIERPINSKI GASKETS

DERONG KONG AND WENXIA LI

Abstract. Given $\beta \in (1,2)$, the fat Sierpinski gasket $\mathcal{S}_\beta$ is defined by

$$\mathcal{S}_\beta := \left\{ \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} : d_i \in \{(0,0),(1,0),(0,1)\} \text{ for all } i \geq 1 \right\}.$$

Let $\mathcal{U}_\beta$ be the set of points in $\mathcal{S}_\beta$ having a unique β-expansion with respect to the alphabet $\{(0,0),(1,0),(0,1)\}$. In this paper we give a lexicographical characterization of $\mathcal{U}_\beta$, and determine the critical base $\beta_c \approx 1.55263$ such that $\mathcal{U}_\beta$ has positive Hausdorff dimension if and only if $\beta > \beta_c$. We show that β_c is transcendental and is defined in terms of a modified Thue-Morse sequence. When $\beta = \beta_c$, the univoque set $\mathcal{U}_\beta$ is uncountable but has zero Hausdorff dimension. When $\beta < \beta_c$, the univoque set $\mathcal{U}_\beta$ is at most countable. This generalizes the one dimensional result of Glendinning and Sidorov (2001) for the unique β-expansions to the fat Sierpinski gaskets.

(D. Kong) Mathematical Institute, University of Leiden, PO Box 9512, 2300 RA Leiden, The Netherlands

E-mail address: `d.kong@math.leidenuniv.nl`

(W. Li) Department of Mathematics, Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200062, People's Republic of China

E-mail address: `wxli@math.ecnu.edu.cn`

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Multi-base Representations and their Minimal Hamming Weight

Daniel Krenn* Vorapong Suppakitpaisarn† Stephan Wagner‡

Given a finite set of bases $b_1, b_2, \ldots, b_r$ (integers greater than 1), a multi-base representation uses, in analogy to a standard base-b representation, all numbers in the set

$$\mathcal{B} = \{b_1^{\alpha_1} b_2^{\alpha_2} \cdots b_r^{\alpha_r} \mid \alpha_1,\, \alpha_2,\, \ldots,\, \alpha_r \in \mathbb{N}_0\}.$$

More precisely, a multi-base representation of a positive integer n is a representation of the form

$$n = \sum_{B \in \mathcal{B}} d_B B, \tag{1}$$

where the digits d_B are taken from a fixed finite set containing 0. Note that we obtain the standard base-b representation if $r = 1$, $b_1 = b$ and $D = \{0, 1, \ldots, b-1\}$.

The *Hamming weight* of a representation (1) is the number of nonzero terms in the sum. The Hamming weight is a measure of how efficient a certain representation is.

The Hamming weight of single-base representations has been thoroughly studied, not only in the case of the standard set $\{0, 1, \ldots, b-1\}$ of digits, but also for more general types of digit sets. Both the worst case (maximum) and the average order of magnitude of the Hamming weight are $\log n$. We investigate the Hamming weight of multi-base representations; it can be reduced—even in the worst case—by using multi-base representations, albeit only by a small amount. Perhaps surprisingly, the order of magnitude is independent of r (provided only that $r \geq 2$), the set of bases and the set of digits: it is always $\log n / (\log \log n)$.

We present the following theorem:

*`daniel.krenn@aau.at`, Alpen-Adria-Universität Klagenfurt, Austria. Daniel Krenn is supported by the Austrian Science Fund (FWF): P 28466-N35.

†`vorapong@is.s.u-tokyo.ac.jp`, The University of Tokyo, Japan.

‡`swagner@sun.ac.za`, Stellenbosch University, South Africa.

Theorem. *Suppose that $r \geq 2$, and that the multiplicatively independent bases b_1, b_2, $\ldots$, b_r and the digit set are such that every positive integer n has a representation of the form* (1)*. There exist two positive constants K_1 and K_2 depending on the bases and the digit set such that the following hold:*

(U) For all integers $n > 2$, there exists a representation of the form (1) *with Hamming weight at most $K_1 \frac{\log n}{\log \log n}$.*

(L) For infinitely many positive integers n, there is no representation of the form (1) *whose Hamming weight is less than $K_2 \frac{\log n}{\log \log n}$.*

The upper bound (U) was shown by Dimitrov, Jullien and Miller [1] in the case that the b_1, b_2 , $\ldots$, b_r are distinct primes. The proof is based on an analysis of the natural greedy algorithm and based on a results by Tijdeman [2] from Diophantine approximation. We present a variant of this proof under the slightly more general condition that the bases b_1, b_2, $\ldots$, b_r are multiplicatively independent.

The lower bound (L) showing that the order of the Greedy algorithm is best possible—therefore showing that the minimal Hamming weight admits the same order—is based on a counting argument. In the talk, the details will be revealed. An alternative approach using communication complexity will also be discussed.

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Infinite families of number systems

Jakub Krásenský and Attila Kovács

Abstract In the context of number systems in lattices, we find a broad class of radices such that for every one of them there exists an infinite family of digit sets where the digits get arbitrarily far away from the origin.

1 Extended abstract

The concept of number systems in lattices, pioneered by Vince [5], is a straightforward generalization of number systems in the ring of integers of a given algebraic number field. The systematic research was initiated by Kátai and continued by Gilbert, B. Kovács and Pethő. Since every lattice can be transformed into $\mathbb{Z}^d$ by a basis transformation, we develop our notions in the convenient setting of an integer lattice.

Let a regular matrix (*radix*) $L \in \mathbb{Z}^{d \times d}$ and a finite set of *digits* $0 \in D \subset \mathbb{Z}^d$ be given. The pair (L, D) is called a number system (*GNS*) in $\mathbb{Z}^d$ if every non-zero element $z \in \mathbb{Z}^d$ has a unique representation of the form

$$z = \sum_{k=0}^{N} L^k a_k, \qquad \text{where } N \in \mathbb{N}_0,\ a_k \in D,\ a_N \neq 0.$$

The necessary conditions for (L, D) being a number system is that D must be a complete residue system modulo L, the radix L must be expansive (i.e. $\rho(L^{-1}) < 1$) and the "unit condition" $\det(L - I) \neq \pm 1$ must be satisfied. Also, for any given (L, D) there are algorithms to decide whether it is a GNS or not, however, with unknown algorithmic complexity.

Jakub Krásenský
Czech Technical University in Prague, Czech Republic, e-mail: krasejak@fjfi.cvut.cz

Attila Kovács
Eötvös Loránd University, Budapest, Hungary, e-mail: attila.kovacs@inf.elte.hu

The main question is giving sufficient conditions on the operator L such that there exists at least one GNS with this radix. In special cases of quadratic number fields this problem was examined e.g. by Steidl [4] and Kátai [2]. The following theorem generalizes the previous result [1]:

Theorem 1 (Germán, Kovács). *If for $L \in \mathbb{Z}^{d\times d}$ the spectral radius of its inverse satisfies $\rho(L^{-1}) < 1/2$, then there always exists a digit set D such that (L,D) is a GNS in $\mathbb{Z}^d$.*

The suitable digit set can be found as follows: take any vector norm on $\mathbb{R}^d$ such that the induced matrix norm satisfies $\|L^{-1}\| < 1/2$. Then we create the so-called *dense digit set* by taking the smallest representative (with respect to this norm) from each congruence class modulo L.

All known number system constructions use digit sets whose elements are very close to the origin (canonical, symmetric, adjoint). We examined the following question: is it possible to create number systems with "sparse" digit sets, i.e. systems where the digits are arbitrarily far away from the origin? We answer this question affirmatively for a broad class of radices. Hence, it is possible to construct an infinite number of different GNSs:

Theorem 2. *Let L be a regular $d \times d$ integer matrix. Suppose further that*

1. *$\rho(L^{-1}) < 1/2$, i.e. all eigenvalues of L are bigger than 2;*
2. *L is diagonalizable over $\mathbb{C}$;*
3. *the characteristic polynomial of L is f^k for some $k \in \mathbb{N}$ and f is irreducible over $\mathbb{Q}$;*
4. *either $d \geq 3$ or L has real eigenvalues and $d = 2$.*

Then we can describe a sequence of digit sets D_n such that for every n the pair (L,D_n) is a GNS, and for any given finite subset S of the lattice, $0 \notin S$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, the set D_n doesn't use any of the elements of S.

We also give several other sets of conditions which enable this construction of "arbitrarily sparse" alphabets, and show that the these conditions are equivalent to the previous ones.

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Ito α-continued fractions and matching

Niels Langeveld

In this talk, we look at Ito α-continued fractions and matching. Matching tells us when the entropy as a function of α is increasing constant or decreasing. We will look at differences and similarities between the case of Ito-α continued fractions and other continued fraction algorithms.

Let $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ be defined by

$$T_\alpha(x) = \begin{cases} S(x) - \lfloor S(x) + 1 - \alpha \rfloor & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases} \tag{1}$$

Different choices of S in formula (1) give rise to different generalizations of the classical continued fraction algorithms:

(N) for $S(x) = \frac{1}{|x|}$ one gets the α-continued fractions first studied by H. Nakada [8].

(KU) for $S(x) = -\frac{1}{x}$ one gets the α-continued fractions first studied by S. Katok and I. Ugarcovici [5].

(IT) for $S(x) = \frac{1}{x}$ one gets the α-continued fractions first studied by S. Ito and S. Tanaka [4].

As in cases (N) and (KU), also for Ito-Tanaka continued fractions the *matching property* will play a central role; a parameter $\alpha \in [0, 1]$ satisfies the *matching condition* with *matching exponents* M, N if

$$T_\alpha^N(\alpha) = T_\alpha^M(\alpha - 1). \tag{2}$$

For matching we will use the following definition:

Definition 1 (Matching)**.** *Let $J \subset [0, 1]$ be an interval with non-empty interior. We say that J is a matching interval with exponents N, M if*

(i) condition (2) holds for every $\alpha \in J$;

(ii) $\{\alpha - 1, \alpha\} \cap PM_\alpha = \emptyset$, where PM_α is the pre-matching set *defined by*

$$PM_\alpha = \{T_\alpha^j(\alpha), 0 < j < N\} \cup \{T_\alpha^i(\alpha - 1), 0 < i < M\};$$

(iii) condition (2) does not hold if we decrease the both exponents by 1, moreover J is not properly contained in any larger interval on which condition (2) holds.

The difference $\Delta := M - N$ is called matching index.

We call matching set *the set A obtained by the union of all matching intervals; its complement will be called* bifurcation set *and will be denoted by $\mathcal{E}$.*

Using a shadowing argument, one can show that for all $\alpha \in \mathbb{Q} \cap (0,1)$ there is a matching interval J such that $\alpha \in J$ in the case of (N) and (KU). For Ito α continued fractions this is not true. For some rationals we will find that, even though 2 holds, there is no matching interval that they are contained in. In fact, interesting behaviour can be observed around these rationals. Let Q_b the set of such rationals. We have the following theorem.

Theorem 0.1. *Let $b \in Q_b \cap (0, \frac{1}{2})$. Then there exists sequences of intervals $(I_n), (J_n), (K_n), (L_n)$, $n \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have*

1. *$I_n < J_n < I_{n+1} < J_{n+1} < b$ and $b < K_{n+1} < L_{n+1} < K_n < L_n$*
2. *For all $n \in \mathbb{N}$ the entropy is constant on I_n and K_n and is increasing on J_n and L_n.*
3. *for all $\delta > 0$ we have that there is an n such that*
   - $(b - \delta, b) \cap I_n \neq \emptyset$
   - $(b - \delta, b) \cap J_n \neq \emptyset$
   - $(b, b + \delta) \cap K_n \neq \emptyset$
   - $(b, b + \delta) \cap L_n \neq \emptyset$
4. *we can find an increasing sequence $(c_n)_{n \geq 1} \subset Q_b$ and a decreasing sequence $(d_n)_{n \geq 1} \subset Q_b$ such that $\lim_{n \to \infty} c_n = \lim_{n \to \infty} d_n = b$.*

In the talk we will construct such sequences explicitly. We will also relate this to an open dynamical system.

This is joint work with Carlo Carminati, Hitoshi Nakada and Wolfgang Steiner.

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THE SUM-OF-DIGITS FUNCTION OF LINEARLY RECURRENT NUMBER SYSTEMS AND ALMOST PRIMES

MANFRED G. MADRITSCH

This is joint work with Jörg Thuswaldner from University of Leoben and Mario Weitzer from Graz University of Technology.

A linear recurrent number system is a generalization of the q-adic number system. In particular, we replace the sequence of powers of q by a linear recurrent sequence $G_{k+d} = a_1 G_{k+d-1} + \cdot + a_d G_k$ for $k \geq 0$. Under some mild conditions for every positive integer n we have a representation of the form

$$n = \sum_{j=0}^{k} \varepsilon_j(n) G_j.$$

The q-adic number system corresponds to the linear recursion $G_{k+1} = qG_k$ and $G_0 = 1$. The first example of a real generalization is due to Zeckendorf who showed that the Fibonacci sequence $G_0 = 1$, $G_1 = 2$, $G_{k+2} = G_{k+1} + G_k$ for $k \geq 0$ yields a representation for each positive integer. This is unique if we additionally suppose that no two consecutive ones exist in the representation.

In the present talk we investigate the representation of primes and almost primes in linear recurrent number systems. We start by showing the different results due to Fouvry, Mauduit and Rivat [2–4]in the case of q-adic number systems. Then we shed some light on their main tools and techniques. The hearth of our considerations is the following Bombieri-Vinogradov type result

$$\sum_{q < x^{\vartheta - \varepsilon}} \max_{y<x} \max_{1 \leq a \leq q} \left| \sum_{\substack{n<y, s_G(n) \equiv b \bmod d \\ n \equiv b \bmod q}} 1 - \frac{1}{q} \sum_{n<y, s_G(n) \equiv b \bmod d} 1 \right| \ll x (\log 2x)^{-A},$$

which we establish under the assumption that $a_1 \geq 30$. This lower bound is due to numerical estimation.

With this tool in hand we aim for lower bounds on the sets of primes and almost primes such that

$$|\{n \leq x \colon s_G(n) \equiv b \bmod d, n = p_1 \text{ or } n = p_1 p_2\}| \gg \frac{x}{\log x}.$$

Finally we want to discuss further related problems like lower estimates for polynomials instead of almost primes as have been established by Dartyge, Stoll and Tenenbaum [1, 5]

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Manfred Madritsch

1. Université de Lorraine, Institut Elie Cartan de Lorraine, UMR 7502, Vandoeuvre-lès-Nancy, F-54506, France;

2. CNRS, Institut Elie Cartan de Lorraine, UMR 7502, Vandoeuvre-lès-Nancy, F-54506, France

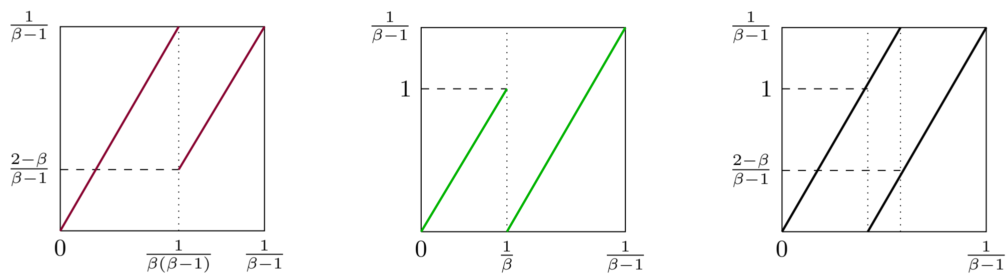
E-mail address: `manfred.madritsch@univ-lorraine.fr`

CONSTRUCTING INVARIANT DENSITIES FOR RANDOM SYSTEMS

MARTA MAGGIONI

This is a joint work with Charlene Kalle. Ergodic theory is convenient for understanding the global behaviour of series expansions of numbers in a given interval. Indeed, series expansions can be generated by iterations of an appropriate map T, where the digits are defined by recursive relations. Generally, most of the maps studied to generate number expansions are Lasota-Yorke maps ([LY73]), i.e. expanding, piecewise C^2-transformations on the interval. These maps have been intensively studied. For instance, a number of articles have been published on absolutely continuous invariant measures of these type of transformations. These measures allow to obtain properties of the related series expansions, such as digits frequency. A prime example is the binary expansions, where the map $T(x) = 2x \mod 1$ associates with each point x of the unit interval $[0, 1)$ an infinite sequence of 0's and 1's. Another classical example, although more complicated, is provided by the greedy and lazy β-transformations, for a non integer $\beta > 1$. See [Rén57, Par60, Gel59, DK10, EJK90, Sid03, DdV07] for example.

The deterministic case becomes even more interesting when replaced with a random setting. In this context, instead of a single map T, a family of maps $T_1, ..., T_r$ is considered from which one is selected at each iteration according to a probabilistic regime. Also random systems are used to generate number expansions. The random β-expansions introduced in [DK03] by Dajani and Kraaikamp provides an example, see Figure 1. It uses random combinations of two piecewise linear maps with constant slope $\beta > 1$. See [DdV05, DK13, Kem14] for example.



(a) T_0 (b) T_1 (c) T

FIGURE 1. In (a) we see the lazy β-transformation, in (b) the greedy β-transformation and in (c) we see them combined. Whether or not $1 > \frac{2-\beta}{\beta-1}$ depends on the chosen value of β.

In [DdV07] it was shown that these maps have a unique absolutely continuous invariant measure. In [Kem14] Kempton gave a formula for the invariant density of the random β-transformation if one chooses the maps according to the uniform Bernoulli regime, and very recently Suzuki ([Suz17]) extended these results to include the non-uniform Bernoulli regime as well.

Except for this specific case, not much is known about the absolutely continuous invariant measures of general random maps. In [Pel84] Pelikan gave sufficient conditions under which a random system, using a finite number of Lasota-Yorke maps, has absolutely continuous invariant measures. He also discussed the possible number of ergodic components. But no explicit expressions for invariant densities were given.

We therefore study invariant densities for any random system of piecewise linear maps that are expanding on average. More precisely, we provide a procedure to construct an explicit formula for the density of an absolutely continuous invariant measure. We obtain this result by generalising the method from [Kop90], valid in the deterministic setting.

For a given random system T we construct a fundamental matrix M. We prove the existence of a non-trivial solution for the matrix equation $M\gamma = 0$ and we relate each solution γ to the density of an absolutely continuous invariant measure h_γ of the system T. We apply the procedure to different examples. We generalise the results from [Kem14] and [Suz17] regarding the expression for the invariant density for the random β-transformation. Moreover, we study a system that is not everywhere expanding, but is expanding on average, by considering a random combination of the greedy β-transformation and the intermittent (α, β)-transformation introduced in [DHK09], see Figure 2. Last, we study another system that has different slopes, namely a random Lüroth map with a hole, see Figure 3.

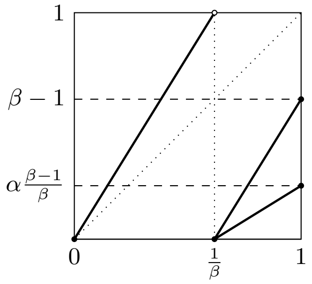
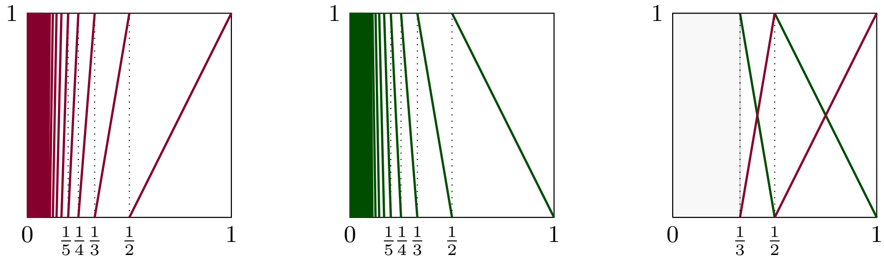


FIGURE 2. The random (α, β)-transformation.



(a) T_L (b) T_A (c) T

FIGURE 3. In (a) we see the Lüroth map and in (b) the alternating Lüroth map. (c) shows the open random system system T consisting of random combinations of T_L and T_A restricted to the interval $\left[\frac{1}{3}, 1\right]$.

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(Marta Maggioni) Mathematisch Instituut, Leiden University, Niels Bohrweg 1, 2333CA Leiden, The Netherlands

E-mail address, Marta Maggioni: `m.maggioni@math.leidenuniv.nl`

SOME COMPLEXITY RESULTS IN THE THEORY OF NORMAL NUMBERS

D. AIREY, S. JACKSON, AND B. MANCE

For a real number r, define real functions π_r and σ_r by $\pi_r(x) = rx$ and $\sigma_r(x) = r + x$. We let $\mathcal{N}(b)$ denote the set of reals x which are normal to base b. We let

$$\mathcal{N}^{\perp}(b) = \{y \colon \forall x \in \mathcal{N}(b)\ \sigma_y(x) \in \mathcal{N}(b)\}.$$

Normality preserving functions. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. We say that f *preserves b-normality* if $f(\mathcal{N}(b)) \subseteq \mathcal{N}(b))$. We can make a similar definition for preserving normality with respect to the regular continued fraction expansion, β-expansions, Cantor series expansions, the Lüroth series expansion, etc.

Several authors have studied b-normality preserving functions. They naturally arise in H. Furstenberg's work on disjointness in ergodic theory[10]. V. N. Agafonov [1], T. Kamae [12], T. Kamae and B. Weiss [13], and W. Merkle and J. Reimann [18] studied b-normality preserving selection rules. The situation for continued fractions is very different. Let $[a_1, a_2, a_3, \ldots]$ be normal with respect to the continued fraction expansion. B. Heersink and J. Vandehey [11] recently proved that for any integers $m \geq 2, k \geq 1$, the continued fraction $[a_k, a_{m+k}, a_{2m+k}, a_{3m+k}, \ldots]$ is never normal with respect to the continued fraction expansion.

In 1949 D. D. Wall proved in his Ph.D. thesis [21] that for non-zero rational r the function π_r is b-normality preserving for all b and that the function σ_r is b-normality preserving for all b whenever r is rational. These results were also independently proven by K. T. Chang in 1976 [8]. D. D. Wall's method relies on the well known characterization that a real number x is normal in base b if and only if the sequence $(b^n x)$ is uniformly distributed mod 1[16].

D. Doty, J. H. Lutz, and S. Nandakumar took a substantially different approach from D. D. Wall and strengthened his result. They proved in [9] that for every real number x and every non-zero rational number r the b-ary expansions of $x, \pi_r(x)$, and $\sigma_r(x)$ all have the same finite-state dimension and the same finite-state strong dimension. It follows that π_r and σ_r preserve b-normality. It should be noted that their proof uses different methods from those used by D. D. Wall and is unlikely to be proven using similar machinery.

G. Rauzy obtained a complete characterization of $\mathcal{N}^{\perp}(b)$ in [19]. M. Bernay used this characterization to prove that Σ_b has zero Hausdorff dimension [6]. One of the main results of this paper, stated in Corollary 3, is to obtain an exact determination of the descriptive set theoretic complexity of $\mathcal{N}^{\perp}(b)$.

M. Mendés France asked in [17] if the function π_r preserves simple normality with respect to the regular continued fraction for every non-zero rational r. This was recently settled by J. Vandehey [20] who showed that $\frac{ax+b}{cx+d}$ is normal with respect to the continued fraction when x is normal with respect to the continued fraction expansion and integers $a, b, c,$and d satisfy $ad - bc \neq 0$. Work was also done on the normality preserving properties of the functions π_r and σ_r for the Cantor series expansions by the first and third author in [2] and additonally with J. Vandehey in [3]. However, these functions are not well understood in this context.

Descriptive Complexity. In any topological space X, the collection of Borel sets $\mathcal{B}(X)$ is the smallest σ-algebra containing the open sets. They are stratified into levels, the Borel hierarchy, by defining $\mathbf{\Sigma}^0_1$ = the open sets, $\mathbf{\Pi}^0_1 = \neg\mathbf{\Sigma}^0_1 = \{X - A \colon A \in \mathbf{\Sigma}^0_1\}$ = the closed sets, and for $\alpha < \omega_1$ we let $\mathbf{\Sigma}^0_\alpha$ be the collection of countable unions $A = \bigcup_n A_n$ where each $A_n \in \mathbf{\Pi}^0_{\alpha_n}$ for some $\alpha_n < \alpha$. We also let $\mathbf{\Pi}^0_\alpha = \neg\mathbf{\Sigma}^0_\alpha$. Alternatively, $A \in \mathbf{\Pi}^0_\alpha$ if $A = \bigcap_n A_n$ where $A_n \in \mathbf{\Sigma}^0_{\alpha_n}$ where each $\alpha_n < \alpha$. We also set $\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Sigma}^0_\alpha$, in particular $\mathbf{\Delta}^0_1$ is the collection of clopen sets. For any topological space, $\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$. All of the collections $\mathbf{\Delta}^0_\alpha$, $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ are pointclasses, that is, they are closed under inverse images of continuous functions. A basic fact (see [14]) is that for any uncountable Polish space X, there is no collapse in the levels of the Borel hierarchy, that is, all the various pointclasses $\mathbf{\Delta}^0_\alpha$, $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, for $\alpha < \omega_1$, are all distinct. Thus, these levels of the Borel hierarch can be used to calibrate the descriptive complexity of a set. We say a set $A \subseteq X$ is $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) *hard* if $A \notin \mathbf{\Pi}^0_\alpha$ (resp. $A \notin \mathbf{\Sigma}^0_\alpha$). This says A is "no simpler" than a $\mathbf{\Sigma}^0_\alpha$ set. We say A is $\mathbf{\Sigma}^0_\alpha$-*complete* if $A \in \mathbf{\Sigma}^0_\alpha - \mathbf{\Pi}^0_\alpha$, that is, $A \in \mathbf{\Sigma}^0_\alpha$ and A is $\mathbf{\Sigma}^0_\alpha$ hard. This says A is exactly at the complexity level $\mathbf{\Sigma}^0_\alpha$. Likewise, A is $\mathbf{\Pi}^0_\alpha$-complete if $A \in \mathbf{\Pi}^0_\alpha - \mathbf{\Sigma}^0_\alpha$.

A fundamental result of Suslin (see [14]) says that in any Polish space $\mathcal{B}(X) = \mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$, where $\mathbf{\Pi}^1_1 = \neg\mathbf{\Sigma}^1_1$, and $\mathbf{\Sigma}^1_1$ is the pointclass of continuous images of Borel sets. Equivalently, $A \in \mathbf{\Sigma}^1_1$ iff A can be written as $x \in a \leftrightarrow \exists y\ (x, y) \in B$ where $B \subseteq X \times Y$ is Borel (for some Polish space Y). Similarly, $A \in \mathbf{\Pi}^1_1$ iff it is of the form $x \in A \leftrightarrow \forall y\ (x, y) \in B$ for a Borel B. The $\mathbf{\Sigma}^1_1$ sets are also called the *analytic* sets, and $\mathbf{\Pi}^1_1$ the *co-analytic sets*. We also have $\mathbf{\Sigma}^1_1 \neq \mathbf{\Pi}^1_1$ for any uncountable Polish space.

H. Ki and T. Linton [15] proved that the set $\mathcal{N}(b)$ is $\mathbf{\Pi}^0_3(\mathbb{R})$-complete. Further work was done by V. Becher, P. A. Heiber, and T. A. Slaman [4] who settled a conjecture of A. S. Kechris by showing that the set of absolutely normal numbers is $\mathbf{\Pi}^0_3(\mathbb{R})$-complete. Furthermore, V. Becher and T. A. Slaman [5] proved that the set of numbers normal in at least one base is $\mathbf{\Sigma}^0_4(\mathbb{R})$-complete.

K. Beros considered sets involving normal numbers in the difference heirarchy in [7]. Let $\mathcal{N}_k(b)$ be the set of numbers normal of order k in base b. He proved that for $b \geq 2$ and $s > r \geq 1$, the set $\mathcal{N}_r(b) \backslash \mathcal{N}_s(b)$ is $\mathcal{D}_2(\mathbf{\Pi}^0_3)$-complete (see [14] for a definition of the difference hierarchy). Additionally, the set $\bigcup_k \mathcal{N}_{2k+1}(2) \backslash \mathcal{N}_{2k+2}(2)$ is shown to be $\mathcal{D}_\omega(\mathbf{\Pi}^0_3)$-complete.

Results. We are interested in determining the exact descriptive set theoretic complexity of $\mathcal{N}^\perp(b)$ and some related sets. The definition of $\mathcal{N}^\perp(b)$ shows that $\mathcal{N}^\perp(b)$ is $\mathbf{\Pi}^1_1$, since it involves a universal quantification. It is not immediately clear if $\mathcal{N}^\perp(b)$ is a Borel set, but this in fact follows from a result of Rauzy. Specifically, Rauzy [19] characterized $\mathcal{N}^\perp(b)$ in terms of an entropy-like condition called the *noise*. We recall this condition and associated notation from [19]. For any positive integer length ℓ, let $\mathcal{E}_\ell$ denote the set of functions from b^ℓ to b. We call an $E \in \mathcal{E}_\ell$ a *block function* of width ℓ. As in [19] we set, for $x \in \mathbb{R}$,

$$\beta_\ell(x, N) = \inf_{E \in \mathcal{E}_\ell} \frac{1}{N} \sum_{n<N} \inf\{1, |c_n - E(c_{n+1}, \ldots, c_{n+\ell})|\},$$

where $c_0, c_1, \dots$ is the (fractional part) of the base b expansion of x.
We also let for $E \in \mathcal{E}$

$$\beta_E(x,N) = \frac{1}{N} \sum_{n<N} \inf\{1, |c_n - E(c_{n+1}, \dots, c_{n+\ell})|\}.$$

We then define the lower and upper noises $\beta^-(x)$, $\beta^+(x)$ of x by:

$$\beta^-(x) = \lim_{\ell\to\infty} \beta^-_\ell(x),$$

where

$$\beta^-_\ell(x) = \liminf_{N\to\infty} \beta_\ell(x,N).$$

The upper entropy $\beta^+(x)$ is defined similarly using

$$\beta^+(x) = \lim_{\ell\to\infty} \beta^+_\ell(x)$$

where

$$\beta^+_\ell(x) = \limsup_{N\to\infty} \beta_\ell(x,N).$$

For a fixed $E \in \mathcal{E}$ we also let

$$\beta^-_E(x) = \liminf_{N\to\infty} \beta_E(x,N),$$

and similarly for $\beta^+_E(x)$.

Rauzy showed that $x \in \mathcal{N}(b)$ iff it has the maximal possible noise in that $\beta^-(x) = \frac{b-1}{b}$. Furthermore, $x \in \mathcal{N}^\perp(b)$ iff it has minmal possible noise in that $\beta^+(x) = 0$.

It is therefore natural to ask for any $s \in [0, \frac{b-1}{b}]$, what are the complexities of the lower and upper noise sets associated to s. Specifically, we introduce the following four sets.

Definition 1. Let $s \in [0, \frac{b-1}{b}]$. Let

$$\begin{aligned} A_1(s) &= \{x \colon \beta^-(x) \le s\}, \quad A_2(s) = \{x \colon \beta^-(x) \ge s\} \\ A_3(s) &= \{x \colon \beta^+(x) \le s\}, \quad A_4(s) = \{x \colon \beta^+(x) \ge s\} \end{aligned} \tag{1}$$

Finally, we let

$$\begin{aligned} L(s) &= A_1(s) \cap A_2(s) = \{x \colon \beta^-(x) = s\} \\ U(s) &= A_3(s) \cap A_4(s) = \{x \colon \beta^+(x) = s\}. \end{aligned}$$

Thus, $\mathcal{N}(b) = L(\frac{b-1}{b})$, and $\mathcal{N}^\perp(b) = U(0)$. The Ki-Linton result shows that $\mathcal{N}(b)$, and thus $L(\frac{b-1}{b})$ is $\mathbf{\Pi}^0_3$-complete for any base b. Recall also the Becher-Slaman result which shows that the set of reals which are normal to some base b forms a $\mathbf{\Sigma}^0_4$-complete set.

We have the following complexity results.

Theorem 2. *For any $s \in [0, \frac{b-1}{b})$, the set $A_1(s)$ is $\mathbf{\Pi}^0_4$-complete and the set $A_3(s)$ is $\mathbf{\Pi}^0_3$-complete. For any $s \in (0, \frac{b-1}{b}]$, the set $A_2(s)$ is $\mathbf{\Pi}^0_3$-complete, and the set $A_4(s)$ is $\mathbf{\Pi}^0_2$-complete. For $s \in (0, \frac{b-1}{b})$, the set $L(s)$ is $\mathbf{\Pi}^0_4$-complete, and the set $U(s)$ is $\mathbf{\Pi}^0_3$-complete.*

As a corollary we obtain the Ki-Linton result as well as the determination of the exact complexity of $\mathcal{N}^\perp(b)$.

Corollary 3. The sets $\mathcal{N}(b)$ and $\mathcal{N}^\perp(b)$ are both $\mathbf{\Pi}^0_3$-complete.

We remark on the significance of complexity classifications such Theorem 2. Aside from the intrinsic interest to descriptive set theory, results of this form can be viewed as ruling out the existence of further theorems which would reduce the complexity of the sets. For example, Rauzy's theorem reduces the complexity of $\mathcal{N}^{\perp}(b)$ from $\mathbf{\Pi}_1^1$ to $\mathbf{\Pi}_3^0$. The fact that $A_3(0)$ is $\mathbf{\Pi}_3^0$-complete tells us that there cannot be other theorems which result in a yet simpler characterization of $\mathcal{N}^{\perp}(b)$.

Lastly, we wish to approximate the Hausdorff dimension of the sets $A_i(s), U(s)$, and $L(s)$. Put $H(s) = -s \log s - (1-s)\log(1-s)$.

Theorem 4. *For* $s \in \left[0, \frac{b-1}{b}\right]$ *we have*

$$\begin{aligned}
\dim_H(A_1(s)) &= 1\\
\dim_H(A_2(s)) &= 1\\
\frac{1}{\log b}H(s) + \frac{\log(b-1)}{\log b}s \le \dim_H(A_3(s)) &\le \frac{1}{\log b}H(s) + s\\
\dim_H(A_4(s)) &= 1.
\end{aligned}$$

Furthermore

$$\begin{aligned}
\frac{1}{\log b}H(s) + \frac{\log(b-1)}{\log b}s \le \dim_H(U(s)) &\le \frac{1}{\log b}H(s) + s\\
\dim_H(L(s)) &= 1.
\end{aligned}$$

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MÖBIUS ORTHOGONALITY FOR AUTOMATIC SEQUENCES AND BEYOND

CLEMENS MÜLLNER

The presented results are partly joint work with Michael Drmota and Lukas Spiegelhofer.

This talk focuses on two different methods to show that fixed points of substitutions can not correlate with the Möbius function. First we recall the case of substitutions of fixed length. Fixed points of substitutions of fixed length correspond to automatic sequences and the author proved that any automatic sequence is orthogonal to the Möbius function. This result relies on a new structural result for deterministic finite automata and uses a method developed by Mauduit and Rivat.

In the case of non-constant length substitutions, very little is known and we focus only on one particular case. It was recently shown by Drmota, Müllner and Spiegelhofer that the sequence $(-1)^{s_\varphi(n)}$ is asymptotically orthogonal to all bounded multiplicative functions, where s_φ denotes the Zeckendorf sum-of-digits function. In particular we have $\sum_{n<N}(-1)^{s_\varphi(n)}\mu(n) = o(N)$, that is, this sequence is orthogonal to the Möbius function.

We use the Katai - Bourgain - Sarnak - Ziegler criterion to reduce the problem to estimates of $\sum_{n<N}(-1)^{s_\varphi(pn)+s_\varphi(qn)}$. To analyze such sums we use the concept of quasi-additivity with respect to the Zeckendorf expansion which allows a generating function approach, which was introduced by Kropf and Wagner for integer bases.

ROTATION NUMBER OF CONTRACTED ROTATIONS

Arnaldo Nogueira, *Institut des Mathématiques de Marseille*
Based on a joint work with Michel Laurant.

Let $I = [0, 1)$ be the unit interval.

Definition 1. *Let* $0 < \lambda < 1$ *and* $\delta \in I$. *We call the map defined by*

$$f = f_{\lambda,\delta} : x \in I \mapsto \{\lambda x + \delta\},$$

where the symbol $\{.\}$ *stands for the fractional part, a contracted rotation of* I.

If $\lambda + \delta > 1$, f is a *2-interval piecewise contraction* on the interval I (see Figure 1).

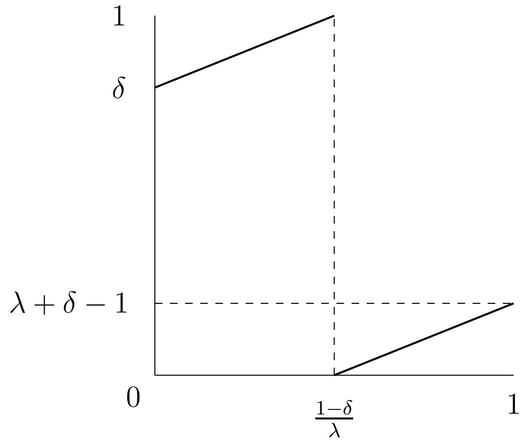

FIGURE 1. A plot of $f_{\lambda,\delta} : I \to I$, where $\lambda + \delta > 1$

Many authors have studied the dynamics of contracted rotations, as a dynamical system or in applications. It is known that every contracted rotation map f has a rotation number $\rho = \rho_{\lambda,\delta}$, satisfying $0 \le \rho < 1$. The goal of this article is to study the value of the rotation number $\rho_{\lambda,\delta}$ according to the diophantine nature of the parameters λ and δ. Applying a classical transcendence result due to J.H. Loxton and A.J. van der Poorten, we prove

Theorem 1. *Let* $0 < \lambda, \delta < 1$ *be algebraic real numbers. Then, the rotation number* $\rho_{\lambda,\delta}$ *is a rational number.*

In view of Theorem 1, a natural problem that arises is to estimate the height of the rational rotation number $\rho_{\lambda,\delta}$ in terms of the algebraic numbers λ and δ. We provide a partial solution for this issue when λ and δ are rational.

Theorem 2. *Let* $\lambda = a/b$ *and* $\delta = r/s$ *be rational numbers with* $0 < \lambda, \delta < 1$. *Assume that* $b > a^\gamma$, *where* $\gamma = \frac{1+\sqrt{5}}{2}$ *denotes the golden ratio. Then, the rotation number* $\rho_{\lambda,\delta}$ *is a rational number* p/q *where*

$$0 \le p < q \le \gamma^{2+\frac{\gamma \log(sb)}{\log b - \gamma \log a}}.$$

Our proofs of Theorems 1 and 2 are based on an arithmetical analysis of formulae giving the rotation number $\rho_{\lambda,\delta}$ in terms of the parameters λ and δ. As far as we are aware, it is in the works of E. J. Ding and P. C. Hemmer and Y. Bugeaud that appears the first complete description of the relations between the parameters λ, δ and the rotation number $\rho_{\lambda,\delta}$. For $0 < \lambda < 1$ fixed, these papers deal with the variation of the rotation number in the one-dimensional family of contracted rotations $f_{\lambda,\delta}$ as δ runs through the interval $[0,1)$. We summarize the results that we need in the following

Theorem 3. *Let $0 < \lambda < 1$ be given. Then the application $\delta \mapsto \rho_{\lambda,\delta}$ is a continuous non decreasing function sending I onto I and satisfying the following properties:*
(i) The rotation number $\rho_{\lambda,\delta}$ vanishes exactly when $0 \leq \delta \leq 1-\lambda$.
(ii) Let $\frac{p}{q}$ be a positive rational number, where $0 < p < q$ are relatively prime integers. Then $\rho_{\lambda,\delta}$ takes the value $\frac{p}{q}$ if and only if δ is located in the interval

$$\frac{1-\lambda}{1-\lambda^q} c\left(\lambda, \frac{p}{q}\right) \leq \delta \leq \frac{1-\lambda}{1-\lambda^q}\left(c\left(\lambda, \frac{p}{q}\right) + \lambda^{q-1} - \lambda^q\right),$$

where

$$c\left(\lambda, \frac{p}{q}\right) = 1 + \sum_{k=1}^{q-2}\left(\left[(k+1)\frac{p}{q}\right] - \left[k\frac{p}{q}\right]\right)\lambda^k$$

and the above sum equals 0 when $q = 2$.
(iii) For every irrational number ρ with $0 < \rho < 1$, there exists one and only one real number δ such that $0 < \delta < 1$ and $\rho_{\lambda,\delta} = \rho$ which is given by the formula

$$\delta = \delta(\lambda, \rho) = (1-\lambda)\left(1 + \sum_{k=1}^{+\infty}\left([(k+1)\rho] - [k\rho]\right)\lambda^k\right). \qquad (1)$$

The proof of Theorem 2 deeply relies on a tree structure, introduced by Y. Bugeaud and J.-P. Conze which is parallel to the classical Stern-Brocot tree of rational numbers. It enables us to handle more easily the complicated intervals occuring in Theorem 3 (ii).

On Schneider's Continued Fraction Map on a Complete Non-Archimedean Field

Speaker: R. Nair (Liverpool)

Abstract : This is joint work with A. Haddley (nee Jaššová) of the University of Liverpool. The purpose of this talk is to calculate the entropy of T. Schneider's continued fraction map, and to show the map has a natural extention which is Bernoulli. Schneider's map is the natural analogue of the Gauss continued fraction map in this setting. The Gauss map is known to have a Bernoulli natural extention with entropy $\frac{\pi^2}{6\log(2)}$. Schneider's map is usually defined on the p-adic field for the rational prime p. In fact we work in a more general setting which we now describe. Let K denote a topological field. By this we mean that the field K is a locally compact group under the addition, with respect to a topology. This ensures that there is a translation invariant Haar measure μ on K, that is unique up to scalar multiplication. In the non-Archimedean examples that concern us, this topology will always be discrete. For an element $a \in K$, we are now able to define its absolute value, as

$$|a| = \frac{\mu(aF)}{\mu(F)},$$

for every μ measureable $F \subseteq K$ of finite positive μ measure. Let $\mathbb{R}_{\geq 0}$ denote the set of non-negative real numbers. An absolute value is a function $|.| : K \to \mathbb{R}_{\geq 0}$ such that (i) $|a| = 0$ if and only if $a = 0$; (ii) $|ab| = |a||b|$ for all $a, b \in K$ and (iii) $|a + b| \leq |a| + |b|$ for all pairs $a, b \in K$. The absolute value just defined gives rise to a metric defined by $d(a, b) = |a - b|$ with $a, b \in K$, whose topology coincides with original topology on the field K.

Topological fields come in two types. The first where (iii) can be replaced by the stronger condition (iii)* $|a + b| \leq \max(|a|, |b|)$ $a, b \in K$, called non-Archimedean fields and fields where (iii)* is not true called Archimedean spaces. In this paper we shall concern ourselves solely with non-Archimedean fields. Another approach to defining a non-Archimedan field is via discrete valuations. Denote the real numbers by $\mathbb{R}$. Let $K^* = K\backslash\{0\}$. A map $v : K^* \to \mathbb{R}$ is a valuation if (a) $v(K^*) \neq \{0\}$; (b) $v(xy) = v(x) + v(y)$ for $x, y \in K$ and (c) $v(x + y) \geq \min\{v(x), v(y)\}$. Two valuations v and cv, for

$c > 0$ a real constant, are called equivalent. We extend v to K formally by letting $v(0) = \infty$. The image $v(K^*)$ is an additive subgroup of $\mathbb{R}$ called the value group of v. If the value group is isomorphic to $\mathbb{Z}$, we say v is a discrete valuation. Here $\mathbb{Z}$ denotes the set of integers. If $v(K^*) = \mathbb{Z}$, we call v a normalised discrete valuation. To our initial absolute value we associate the valuation described as follows. Pick $0 < \alpha < 1$ and write $|a| = \alpha^{v(a)}$, i.e., let $v(a) = \log_\alpha |a|$. Then $v(a)$ is a valuation, an additive version of $|a|$.

Let $v : K^* \to \mathbb{R}$ be a valuation corresponding to the absolute value $|.| : K \to \mathbb{R}_{\geq 0}$. Then

$$\mathcal{O} = \mathcal{O}_v := \{x \in K : v(x) \geq 0\} = \mathcal{O}_K := \{x \in K : |x| \leq 1\}$$

is a ring, called the valuation ring of v and K is its field of fractions. The set of units in $\mathcal{O}$ is $\mathcal{O}^\times = \{x \in K : v(x) = 0\} = \{x \in K : |x| = 1\}$ and $\mathcal{M} = \{x \in K : v(x) > 0\} = \{x \in K : |x| < 1\}$ is an ideal in $\mathcal{O}$. Note $\mathcal{O} = \mathcal{O}^\times \cup \mathcal{M}$. Because $\mathcal{M}$ is a maximal ideal, we know $k = \mathcal{O}/\mathcal{M}$ is a field, called the residue field of v or of K. Henceforth we assume that k is a finite field. Suppose the valuation $v : K^* \to \mathbb{Z}$ is normalised and discrete. Take $\pi \in \mathcal{M}$ such that $v(\pi) = 1$. We call π a uniformiser. Then every $x \in K$ can be written uniquely as $x = u\pi^n$ with $u \in \mathcal{O}^\times$ and $n \in \mathbb{Z}$. In particular every $x \in \mathcal{M}$ can be written uniquely as $x = u\pi^n$ for a unit $u \in \mathcal{O}^\times$ and $n \geq 1$.

We now consider two examples.

a) p-adic numbers : Let $\mathbb{Q}$ denote the rational numbers. For $r = p^{v_p}\frac{u}{v}$ in $\mathbb{Q}$ with u and v coprime to p and each other, let $|r|_p = p^{-v_p}$. Then $d_p(r, r') = |r - r'|_p$ for $r' \in \mathbb{Q}$ defines a metric on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric d_p is a field denoted $\mathbb{Q}_p$ referred to as the p-adic numbers. We also use $\mathbb{Z}_p$ to denote $\{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ – the ring of p-adic integers. It is worth keeping in mind that the metric d_p has the ultrametric property, namely that $d_p(r, r'') \leq \max(d_p(r, r'), d_p(r', r''))$ for all r, r' and $r'' \in \mathbb{Q}_p$. The main characteristics of the field $\mathbb{Q}_p$ that distinguish it from the field $\mathbb{R}$ stem from the ultrametric property. It turns out that $\mathbb{Q}_p$ is a locally compact abelian field and hence comes endowed with a translation invariant Haar measure. In this instance $K = \mathbb{Q}_p$, $\mathcal{O} = \mathbb{Z}_p$, $\mathcal{M} = p\mathbb{Z}$, $\pi = p$ and $k = \mathbb{Z}/p\mathbb{Z}$.

b) The field of formal Laurant series in finite characteristic : Let q be a power of a prime p and let $\mathbb{F}_q$ be the finite field with q elements. Denote by $\mathbb{F}_q[X]$ and $\mathbb{F}_q(X)$ the ring of polynomials with coefficients in $\mathbb{F}_q$ and the quotient field of $\mathbb{F}_q[X]$ respectively. For each $P, Q \in \mathbb{F}_q[X]$ set $|P/Q| := q^{\deg(P)-\deg(Q)}$ where for an element $g \in \mathbb{F}_p[X]$ we have denoted its degree by $\deg(g)$. Let $\mathbb{F}_q((X^{-1}))$ denote the field of formal Laurent series i.e.

$$\mathbb{F}_q((X^{-1})) = \left\{a_n X^n + \cdots + a_0 + a_{-1}X^{-1} + \cdots : n \in \mathbb{Z},\ a_i \in \mathbb{F}_q\right\}.$$

Also $d_q(x,y) = |x-y|$ for $x,y \in \mathbb{F}_q(X)$ defines a metric on $\mathbb{F}_q(X)$. The metric extends to $\mathbb{F}_q((X^{-1}))$ by completion and by implication to its subset $\mathbb{L} = \{x \in \mathbb{F}_q((X^{-1})) : |x| \leq 1\}$. Note that this metric is non-Archimedian since $|x+y| \leq \max(|x|,|y|)$. In this example $K = \mathbb{F}_q((X^{-1}))$, $\mathcal{O} = \mathbb{L}$, $\mathcal{M} = X\mathbb{L}$, $\pi = X$ and $k = \mathbb{L}/X\mathbb{L} = \mathbb{F}_q$.

The only two types of non-Archimedean local fields there are are finite exensions of the field of p-adic numbers for some rational prime p and the field of formal Laurant series over a finite field.

Our primary object of study in this paper is the map $T_v : \mathcal{M} \to \mathcal{M}$ defined by

$$T_v(x) = \frac{\pi^{v(x)}}{x} - b(x)$$

where $b(x)$ denotes the residue class to which $\frac{\pi^{v(x)}}{x}$ belongs in k.

This gives rise to the continued fraction expansion of $x \in \mathcal{M}$ in the form

$$x = \cfrac{\pi^{a_1}}{b_1 + \cfrac{\pi^{a_2}}{b_2 + \cfrac{\pi^{a_3}}{b_3 + \ddots}}}$$

where $b_n \in k^\times, a_n \in \mathbb{N}$ for $n = 1,2,\ldots$. Here $\mathbb{N}$ denotes the set of natural numbers.

The rational approximants to $x \in \mathcal{M}$ arise in a manner similar to that in the case of the real numbers as follows. We suppose $A_0 = b_0, B_0 = 1, A_1 = b_0 b_1 + \pi^{a_1}, B_1 = b_1$. Then set

$$A_n = \pi^{a_n} A_{n-2} + b_n A_{n-1} \text{ and } B_n = \pi^{a_n} B_{n-2} + b_n B_{n-1}$$

for $n \geq 2$. A simple inductive argument, for $n = 1,2,\ldots$ gives

$$A_{n-1}B_n - A_n B_{n-1} = (-1)^n \pi^{a_1+\ldots+a_n}.$$

One can readily check that the map $T_v : \mathcal{M} \to \mathcal{M}$ preserves Haar measure on $\mathcal{M}$. By this we mean, for each Haar measurable set A contained in $\mathcal{M}$ we have $\mu(T_v^{-1}(A)) = \mu(A)$. Here $T_v^{-1}(A) := \{x \in \mathcal{M} : T_v(x) \in A\}$.

In the case where $K = \mathbb{Q}_p$ the map T_v reduces to the original Schneider's continued fracton map T_p, which motivates this whole investigation and is defined as follows. For $x \in p\mathbb{Z}_p$ define the map $T_p : p\mathbb{Z}_p \to p\mathbb{Z}_p$ by

$$T_p(x) = \frac{p^{v(x)}}{x} - \left(\frac{p^{v(x)}}{x} \bmod p\right) = \frac{p^{a(x)}}{x} - b(x)$$

where $v(x)$ is the p-adic valuation of x, $a(x) \in \mathbb{N}$ and $b(x) \in \{1, 2, \ldots, p-1\}$. Then using the continued fraction algorithm for x we get the expansion,

$$x = \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{b_3 + \ddots}}}$$

where $b_n \in \{1, 2, \ldots, p-1\}, a_n \in \mathbb{N}$ for $n = 1, 2, \ldots$.

We now define measure-theoretic entropy. Let $(X, \mathcal{A}, m)$ be a probability space where X is a set, $\mathcal{A}$ is a σ-algebra of its subsets and m is a probability measure. A partition of $(X, \mathcal{A}, m)$ is defined as a denumerable collection of elements $\alpha = \{A_1, A_2, \ldots\}$ of $\mathcal{A}$ such that $A_i \cap A_j = \emptyset$ for all $i, j \in \Lambda$ with $i \neq j$ and $\bigcup_{i \in \Lambda} A_i = X$. Here Λ is a denumerable index set. For a measure-preserving transformation T we have $T^{-1}\alpha = \{T^{-1}A_i | A_i \in \alpha\}$ which is also a denumerable partition. Given partitions $\alpha = \{A_1, A_2, \ldots\}$ and $\beta = \{B_1, B_2, \ldots\}$ we define the join of α and β to be the partition $\alpha \vee \beta = \{A_i \cap B_j | A_i \in \alpha, B_j \in \beta\}$. For a finite partition $\alpha = \{A_1, \ldots, A_n\}$ we define its entropy $H(\alpha) := -\sum_{i=1}^{n} m(A_i) \log m(A_i)$. Let $\mathcal{A}' \subset \mathcal{A}$ be a sub-σ-algebra. Then we define the conditional entropy of α given $\mathcal{A}'$ to be $H(\alpha|\mathcal{A}') := -\sum_{i=1}^{n} m(A_i|\mathcal{A}') \log m(A_i|\mathcal{A}')$. Here $m(A|\mathcal{A}')$ denotes the m-conditional probability of A with respect to the σ-algebra $\mathcal{A}'$. See [**?**] for more details about conditional probability. The entropy of a measure-preserving transformation T relative to a partition α is defined to be

$$h_m(T, \alpha) = \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right)$$

where the limit always exists. The alternative formula for $h_m(T, \alpha)$ which is used for calculating entropy is

$$h_m(T, \alpha) = \lim_{n \to \infty} H\left(\alpha | \bigvee_{i=1}^{n} T^{-i}\alpha\right) = H\left(\alpha | \bigvee_{i=1}^{\infty} T^{-i}\alpha\right).$$

We define the measure-theoretic entropy of T with respect to the measure m to be $h_m(T) = \sup_\alpha h_m(T, \alpha)$. Here the supremum is taken over all finite or countable partitions α from $\mathcal{A}$ with $H(\alpha) < \infty$.

Two measure-preseving transformations (X_1, β_1, m_1, T_1) and (X_2, β_2, m_2, T_2) are said to be isomorphic if there exist sets $M_1 \subseteq X_1$ and $M_2 \subseteq X_2$ with $m_1(M_1) = 1$ and $m_2(M_2) = 1$ such that $T_1(M_1) \subseteq M_1$ and $T_2(M_2) \subseteq M_2$

and such that there exists a map $\phi : M_1 \to M_2$ satisfying $\phi T_1(x) = T_2\phi(x)$ for all $x \in M_1$ and $m_1(\phi^{-1}(A)) = m_2(A)$ for all $A \in \beta_2$. The importance of measure theoretic entropy, is that two dynamical systems with different entropies can not be isomorphic. The following is our first result.

Theorem 0.1 *Let $\mathcal{B}$ denote the Haar σ-algebra restricted to $\mathcal{M}$ and let μ denote Haar measure on $\mathcal{M}$. Then the measure-preserving transformation $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ has measure-theoretic entropy $\frac{\#(k)}{\#(k^\times)} \log(\#(k))$.*

The measure-preserving transformation $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ is known to be ergodic [**3**]. Moreover, in [**2**] it was proved that $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ is exact. The exactness of $(p\mathbb{Z}_p, \mathcal{B}, \mu, T_p)$ implies other weaker properties including mixing, which implies weak-mixing implying ergodicity, all implications being strict. Suppose (Y, α, l) is a probability space, and let $Y_n = (Y, \alpha, l)$ for each $n \in \mathbb{Z}$. Set $(X, \beta, m) = \Pi_{n\in\mathbb{Z}} Y_n$ i.e. the bi-infinite product probability space. For the shift map $\tau(\{x_n\}) = (\{x_{n+1}\})$, the measure preserving transformation (X, β, m, τ) is called the Bernoulli process with state space (Y, α, l). Here $\{x_n\}$ is a bi-infinite sequence of elements of the set Y. Any measure preserving transformation isomorphic to a Bernoulli process will be referred to as Bernoulli. The fundamental fact about Bernoulli processes, famously proved by D. Ornstein, is that Bernoulli processes with the same entropy are isomorphic. To any measure-preserving transformation, (X, β, m, T_0) we can associate another called its natural extension. Originally introduced by V. A. Rokhlin, the natural extension is defined as follows. Set

$$X_{T_0} = \{(x_0, x_1, x_2, \dots) : x_n = T_0(x_{n+1}), x_n \in X, n = 0, 1, 2, \dots\},$$

and let $T : X_{T_0} \to X_{T_0}$ be defined by

$$T((x_0, x_1, \dots,)) = (T_0(x_0), x_0, x_1, \dots,).$$

The map T is $1-1$ on X_{T_0}. If T_0 preserves a measure m, then we can define a measure $\overline{m}$ on X_{T_0}, by defining $\overline{m}$ on the cylinder sets

$$C(A_0, A_1, \dots, A_k) = \{\{x_n\} : x_0 \in A_0, x_1 \in A_1, \dots, x_k \in A_k\}$$

by

$$\overline{m}(C(A_0, A_1, \dots, A_k)) = m(T_0^{-k}(A_0) \cap T_0^{-k+1}(A_1) \cap \dots \cap A_k),$$

for $k \geq 1$. One can check that the transformation $(X_{T_0}, \overline{\beta}, \overline{m}, T_0)$ is measure preserving as a consequence of the measure preservation of the transformation (X, β, m, T_0). Our second theorem is the following.

Theorem 0.2 *Suppose $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ is as in our first theorem. Then the dynamical system $(\mathcal{M}, \mathcal{B}, \mu, T_v)$ has a natural extension that is Bernoulli.*

Our two theorems above together tell us that as a dynamical system, the isomorphism class to which T_v belongs is determined solely by its residue class field. This is irrespective of the characteristic of the underlying global field. For instance for each rational prime p the corresponding Schneider map has entropy $\frac{p}{p-1}\log(p)$, so we know these maps are mutually non-isomorphic. Each of them is however isomorpic to the analogue of the Schneider map on the field of formal power series with coefficient field the finite field of p elements.

As is well known, if you restrict the Gauss map to the rational numbers you get the Euclidean algorithm. If you set $p = 2$ and restrict the Schneider map to the rational numbers what you get is the Binary Euclidean algorithm. This is another way of calculating the highest common factor of two integers, particularly well adapted to efficient implementation on binary machines. The algorithm was first published by Josef Stein but is also attributed to Roland Silver and John Terzian. The algorithm may however be much older. D. Knuth cites a verbal description of the algorithm in the first-century A.D. Chinese text “Chiu Chang Suan Shu ”. See the reference below for more details, including missing definitions.

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Essentially non-normal numbers for Cantor series expansions

Dylan Airey and Bill Mance and Roman Nikiforov

Denote by $N_n^b(B,x)$ the number of times a block B occurs with its starting position no greater than n in the b-ary expansion of x.

A real number x is *normal in base b* if for all k and blocks B in base b of length k, one has

$$\lim_{n\to\infty} \frac{N_n^b(B,x)}{n} = b^{-k}. \tag{1}$$

A number x is *simply normal in base b* if (1) holds for $k = 1$.

Borel introduced normal numbers in 1909 and proved that almost all (in the sense of Lebesgue measure) real numbers are normal in all bases. Obviously that the complement of the set of normal numbers has zero Lebesque measure. But how small is the compliment in fractal and topological sense?

Let consider a subset of set of non-normal numbers for which limit (1) does not exist for any individual digit. Such numbers called essentially non-normal numbers. It was proven by Albeverio, Pratsiovytyi and Torbin in 2005 that this set has full Hausdorff dimension and is of second Baire category. This result was extended for different system of numeration with finite alphabet (Q-expansion, Q^*-expansion) and with infinite alphabet (Q_∞-expansion, I-Q_∞-expansion, Lüroth series expansion). We extend and generalize this result for large class of Cantor series expansion considering numbers for which limit (1) does not exist for any block of digits for all k. Furthermore the result still hold for the set of essentially non-normal numbers whose Cantor series digits are sampled along all arithmetic progressions.

Dylan Airey
Princeton University, Fine Hall, Washington Road, Princeton NJ 08544-1000, USA, e-mail: dairey@math.princeton.edu

Bill Mance
Adam Mickiewicz University in Poznan, Wieniawskiego 1, 61-712 Poznan, Poland, e-mail: william.mance@amu.edu.pl

Roman Nikiforov
Dragomanov National Pedagogical University, Pyrogova St. 9, 01601, Kyiv, Ukraine, e-mail: rnikiforov@gmail.com

The Q-Cantor series expansions, first studied by G. Cantor, are a natural generalization of the b-ary expansions.

Let $\mathbb{N}_k := \mathbb{Z} \cap [k, \infty)$. If $Q \in \mathbb{N}_2^{\mathbb{N}}$, then we say that Q is a *basic sequence*. Given a basic sequence $Q = (q_n)_{n=1}^{\infty}$, the *Q-Cantor series expansion* of a real number $x \in [0,1]$ is the unique expansion of the form

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n} \tag{2}$$

where E_n is in $\{0, 1, \dots, q_n - 1\}$ for $n \geq 1$ with $E_n \neq q_n - 1$ infinitely often.

Let $B = (b_1, b_2, \dots, b_k)$ is a block of digits of length k. Then for block B and a natural number j define

$$I_{Q,j}(B) = \begin{cases} 1, & \text{if } b_1 < q_j, b_2 < q_{j+1}, \dots, b_k < q_{j+k-1}, \\ 0, & \text{otherwise,} \end{cases}$$

and let

$$Q_n(B) = \sum_{j=1}^{n} \frac{I_{Q,j}(B)}{q_j q_{j+1} \cdots q_{j+k-1}}.$$

Let $N_n^Q(B,x)$ denote the number of occurrences of the block B in the digits of the Q-Cantor series expansion of x up to position n.

A real number x is essentially non-normal if for all blocks B such $\lim_{n\to\infty} Q_n(B) = \infty$ the limit

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n(B)}$$

does not exist.

NUMBER SYSTEMS OVER ORDERS
EXTENDED ABSTRACT

ATTILA PETHŐ AND JÖRG THUSWALDNER

In this talk, which is based on our joint paper [23] we introduce a general notion of number system defined over orders of number fields. This generalizes the well-known *number systems* and *canonical number systems* in number fields to relative extensions and allows for the definition of "classes" of digit sets by the use of lattice tilings. It fits in the general framework of *digit systems over commutative rings* defined by Scheicher *et al.* [25].

Before the beginning of the 1990s canonical number systems have been defined as number systems that allow to represent elements of orders (and, in particular, rings of integers) in number fields. After the pioneering work of Knuth [16] and Penney [21], special classes of canonical number systems have been studied by Kátai and Szabó [15], Kátai and Kovács [13, 14], and Gilbert [11], while elements of a general theory are due to Kovács [17] as well as Kovács and Pethő [18, 19]. In 1991 Pethő [22] gave a more general definition of canonical number systems that reads as follows. Let $p \in \mathbb{Z}[x]$ be a monic polynomial and let $\mathcal{D}$ be a complete residue system modulo $p(0)$. The pair $(p, \mathcal{D})$ was called a *number system* if each $a \in \mathbb{Z}[x]$ allows a representation of the form

$$a \equiv d_0 + d_1 x + \cdots + d_{\ell-1} x^{\ell-1} \pmod{p} \qquad (d_0, \ldots, d_{\ell-1} \in \mathcal{D}). \tag{1}$$

If such an expansion exists it is unique if we forbid "leading zeros", *i.e.*, if we demand $d_{\ell-1} \neq 0$ for $a \not\equiv 0 \pmod{p}$ and take the empty expansion for $a \equiv 0 \pmod{p}$. It can be determined algorithmically by the so-called "backward division mapping" (see *e.g.* [1, Section 3] or [25, Lemma 2.5]). Choosing the digit set $\mathcal{D} = \{0, 1, \ldots, |p(0)| - 1\}$, the pair $(p, \mathcal{D})$ was called a *canonical number system*, CNS for short. An overview about the early theory of number systems can be found for instance in Akiyama *et al.* [1] and Brunotte, Huszti, and Pethő [7].

Let $p \in \mathbb{Z}[x]$ and let $\mathcal{D}$ be a complete residue system modulo $p(0)$. With the development of the theory of radix representations it became necessary to notationally distinguish an arbitrary pair $(p, \mathcal{D})$ from a particular pair $(p, \mathcal{D})$ for which each $a \in \mathbb{Z}[x]$ admits a representation of the form (1). Nowadays in the literature an arbitrary pair $(p, \mathcal{D})$ is called *number system* (or *canonical number system* if $\mathcal{D} = \{0, 1, \ldots, |p(0)| - 1\}$), while the fact that each $a \in \mathbb{Z}[x]$ admits a representation of the form (1) is distinguished with the suffix *with finiteness property.* Although there exist many partial results on the characterization of number systems with finiteness property with special emphasis on canonical number systems (see for instance [2, 3, 5, 6, 8, 17, 26, 27]), a complete description of this property seems to be out of reach (although there are fairly complete results for finite field analogs, see for instance [4, 9, 20]).

If $(p,\mathcal{D})$ is a number system and $a \in \mathbb{Z}[x]$ admits a representation of the form (1), we call ℓ the *length* of the representation of a in this number system (for $a \equiv 0 \pmod p$ this length is zero by definition).

In the present talk we generalize the CNS concept in two ways. Firstly, instead of looking at polynomials in $\mathbb{Z}[x]$ we consider polynomials with coefficients in some order $\mathcal{O}$ of a given number field of degree k, and secondly, we consider the sets of digits in a more general but uniform way. Indeed, for each fundamental domain $\mathcal{F}$ of the action of $\mathbb{Z}^k$ on $\mathbb{R}^k$ we define a class of number systems $(p, D_{\mathcal{F}})$ where $\mathcal{F}$ associates a digit set $D_{\mathcal{F}}$ with each polynomial $p \in \mathcal{O}[x]$ in a natural way. Thus for each fundamental domain $\mathcal{F}$ we can define a class $\mathcal{G}_{\mathcal{F}} := \{(p, D_{\mathcal{F}}) \;:\; p \in \mathcal{O}[x]\}$ of number systems whose properties will be studied.

Our main objective will be the investigation of the finiteness property for these number systems. For a given pair $(p,\mathcal{D})$ this property can be checked algorithmically. This makes it possible to prove a strong bound for the length of the representations, provided it exists.

The "dominant condition", a condition for the finiteness property of $(p,\mathcal{D})$ that involves the largeness of the absolute coefficient of p, has been studied for canonical number systems in several versions for instance in Kovács [17, Section 3], Akiyama and Pethő [2, Theorem 2], Scheicher and Thuswaldner [26, Theorem 5.8], and Pethő and Varga [24, Lemma 7.3]. The main difficulty of the generalization of the dominant condition is due to the fact that in $\mathcal{O}$ we do not have a natural ordering, hence, we cannot adapt the methods that were used in the case of integer polynomials. However, by exploiting tiling properties of the fundamental domain $\mathcal{F}$ we are able to overcome this difficulty, and provide a general criterion for the finiteness property that is in the spirit of the dominant condition and can be used in the proofs of our main results. In particular, using this criterion, depending on natural properties of $\mathcal{F}$ we are able to show that $(p(x+m), D_{\mathcal{F}})$ enjoys a finiteness property for each given p provided that m (or $|m|$) is large enough. This forms a generalization of an analogous result of Kovács [17] to this general setting. We also give a converse of this result in showing that $(p(x-m), D_{\mathcal{F}})$ doesn't enjoy the finiteness property for large m if $\mathcal{F}$ has certain properties.

If $p \in \mathbb{Z}[x]$ is irreducible then $\mathbb{Z}[x]/(p)$ is isomorphic to $\mathbb{Z}[\alpha]$ for any root α of p. Thus in this case the finiteness property of $(p,\mathcal{D})$ is easily seen to be equivalent to the fact that each $\gamma \in \mathbb{Z}[\alpha]$ admits a unique expansion of the form

$$\gamma = d_0 + d_1\alpha + \cdots + d_{\ell-1}\alpha^{\ell-1} \tag{2}$$

with analogous conditions on $d_0,\dots,d_{\ell-1} \in \mathcal{D}$ as in (1). In this case we sometimes write $(\alpha,\mathcal{D})$ instead of $(p,\mathcal{D})$. This relates number systems to the problem of *power integral bases* of orders. Recall that the order $\mathcal{O}$ has a power integral basis, if there exists $\alpha \in \mathcal{O}$ such that each $\gamma \in \mathcal{O}$ can be written uniquely in the form

$$\gamma = g_0 + g_1\alpha + \cdots + g_{k-1}\alpha^{k-1}$$

with $g_0,\dots,g_{k-1} \in \mathbb{Z}$. In this case $\mathcal{O}$ is called *monogenic*. The definitions of number system with finiteness property (2) and power integral bases seem similar and indeed there is a strong relation between them. Kovács [17, Section 3] proved that the order $\mathcal{O}$ has a power integral basis if and only if it contains α such that $(\alpha, \{0,\dots,|N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha)|-1\})$ is a CNS with finiteness property. A deep result of Győry [12] states that, up to translation by integers, $\mathcal{O}$ admits finitely many power integral bases and they are effectively computable. Combining this result of Győry with the above mentioned theorem of Kovács [17, Section 3], Kovács and Pethő [18] proved that if $1, \alpha, \ldots, \alpha^{k-1}$ is a power integral basis then, up to finitely many possible exceptions, $(\alpha - m, \mathcal{N}_0(\alpha - m)), m \in \mathbb{Z}$ is a CNS with finiteness property if and only if $m > M(\alpha)$, where $M(\alpha)$ denotes a constant. A good overview over this circle of ideas is provided in the book of Evertse and Győry [10].

Using this theorem we generalize the results of Kovács [17] and of Kovács and Pethő [18] to number systems over orders in algebraic number fields. Our result is not only more general as the earlier ones, but sheds fresh light to the classical case of number systems over $\mathbb{Z}$ too. It turns out that under general conditions in orders of algebraic number fields the power integral bases and the bases of number systems with finiteness condition up to finitely many, effectively computable exceptions coincide. Choosing for example the symmetric digit set, the conditions satisfy and, hence, power integral bases and number systems coincide up to finitely many exceptions. This means that CNS are quite exceptional among number systems.

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Department of Computer Science, University of Debrecen, H-4010 Debrecen, P.O. Box 12, HUNGARY and, University of Ostrava, Faculty of Science, Dvořákova 7, 70103 Ostrava, CZECH REPUBLIC
E-mail address: `Petho.Attila@inf.unideb.hu`

Chair of Mathematics and Statistics, University of Leoben, Franz-Josef-Strasse 18, A-8700 Leoben, AUSTRIA
E-mail address: `Joerg.Thuswaldner@unileoben.ac.at`

FUNDAMENTAL DOMAINS FOR RATIONAL FUNCTION BASED DIGIT SYSTEMS OF FORMAL LAURENT SERIES OVER FINITE FIELDS

M.J.C. LOQUIAS AND A.D.C. RODRIGUEZ*

Institute of Mathematics, College of Science, University of the Philippines Diliman

Abstract. Let $\mathbb{F}$ be a finite field. Digit systems for the residue class ring S of the ring $\mathbb{F}((x^{-1}, y^{-1}))$ of formal Laurent series in two variables over $\mathbb{F}$ modulo a polynomial f in $\mathbb{F}[x, y]$ that is monic in both x and y were studied in [1]. In particular, fundamental domains with respect to both x-digit and y-digit representations that induce a tiling of S were examined. More recently, digit systems with rational base P/Q, where $P, Q \in \mathbb{F}[x]$ are coprime with $\deg P > \deg Q \geq 0$, for the field $\mathbb{F}((x^{-1}))$ were introduced in [2]. These new digit systems may be viewed as for the ring S formed from the polynomial $f = Qy - P$, which is not necessarily monic in y. In this contribution, we obtain fundamental domains with respect to P/Q-digit representations given by such digit systems and investigate some of their properties.

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OPEN MAPS: SMALL AND LARGE HOLES

NIKITA SIDOROV

Let X be a compact manifold, $T : X \to X$ piecewise continuous with positive entropy, and μ a T-invariant probability measure on X. Let H be an open connected subset of X which we treat as a hole. Then the *survivor set* $J(H)$ is the set of all $x \in X$ whose T-orbits avoid H.

In my talk I will concentrate on the following questions:

(i) If $\mu(H)$ is sufficiently small, is it true that the Hausdorff dimension of $J(H)$ is positive?

(ii) If $\mu(H)$ is sufficiently close to 1, is it true that $J(H)$ is countable?

We will see that the answers strongly depend on whether X is one-dimensional. My main examples will be the doubling map and the baker's map.

As it turns out, in the case of the doubling map and $H = (a, b)$, the *critical* holes are closely related to balanced words and Sturmian sequences. For the baker's map the situation is very different, although the Thue-Morse sequence appears to play a significant role as well.

This talk is based on my recent papers with Paul Glendinning (Manchester) and Kevin Hare (Waterloo).

THE LEVEL OF DISTRIBUTION OF THE THUE–MORSE SEQUENCE

LUKAS SPIEGELHOFER

Abstract. The level of distribution of a complex valued sequence b measures how well b behaves on arithmetic progressions $nd+a$. Determining whether a given number θ is a level of distribution for b involves summing a certain error over $d \le D$, where D depends on θ; this error is given by comparing a finite sum of b along $nd+a$ and the expected value of the sum. We prove that the Thue–Morse sequence has level of distribution 1, which is essentially best possible. More precisely, this sequence gives one of the first nontrivial examples of a sequence satisfying an analogue of the Elliott–Halberstam conjecture in prime number theory. In particular, this result improves on the level of distribution 2/3 obtained by Müllner and the author.

Moreover, we show that the subsequence of the Thue–Morse sequence indexed by $\lfloor n^c \rfloor$, where $1 < c < 2$, is simply normal. That is, each of the two symbols appears with asymptotic frequency 1/2 in this subsequence. This result improves on the range $1 < c < 3/2$ obtained by Müllner and the author and closes the gap that appeared when Mauduit and Rivat proved (in particular) that the Thue–Morse sequence along the squares is simply normal. In the proofs, we reduce both problems to an estimate of a certain Gowers uniformity norm of the Thue–Morse sequence similar to that given by Konieczny (2017).

1. Extended abstract

Let $\mathbf{t}$ be the Thue–Morse sequence $\mathbb{N} \to \{\mathtt{0},\mathtt{1}\}$ defined by $\mathbf{t}(n) = \mathtt{0}$ if and only if $2 \mid s(n)$, where $s(n)$ is the binary sum-of-digits function. The main topic of this article is the study of $\mathbf{t}$ along arithmetic progressions and, more generally, along Beatty sequences $\lfloor n\alpha + \beta \rfloor$. We are particularly interested in the error term for *sparse* arithmetic progressions, having large common difference d. This is the subject of our main result.

Theorem 1.1. *Assume that $\varepsilon > 0$. Then*

$$\sum_{1 \le d \le x^{1-\varepsilon}} \max_{0 \le y \le x} \max_{0 \le a < d} \left| \sum_{\substack{0 \le n < y \\ n \equiv a \bmod d}} (-1)^{s(n)} \right| = \mathcal{O}(x^{1-\eta})$$

for some $\eta > 0$ depending on ε.

The formulation of this theorem is an analogue of the Elliott–Halberstam conjecture.

Conjecture (Elliott–Halberstam)**.** *For all real numbers $A > 0$ there exists a constant C such that for all $x \ge 2$*

$$\sum_{1 \le d \le x^{1-\varepsilon}} \max_{\substack{0 \le a < d \\ \gcd(a,d)=1}} \left| \pi(x; d, a) - \frac{\pi(x)}{\varphi(d)} \right| \le Cx(\log x)^{-A}.$$

Here φ denotes Euler's totient function, $\pi(x)$ is the number of primes up to x, and $\pi(x; d, a)$ is the number of such primes that are $\equiv a \bmod d$.

Our result says that the Thue–Morse sequence has *level of distribution* 1, while the Elliott–Halberstam conjecture states that the primes have level of distribution 1. It is known that 1/2 is a level of distribution of the primes (the Bombieri–Vinogradov theorem).

A level of distribution 0.55711 for the sum of digits in base 2 modulo m (and also a level 0.5924 for Thue–Morse) was obtained by Fouvry and Mauduit [2]. Using sieve theory, they applied this result to the study of $s(n) \bmod m$ for positive integers n having at most two prime factors. They also considered [1] the sum of digits in base q modulo m, where $\gcd(m, q-1) = 1$ and obtained the result that the level of distribution approaches 1 as the base q grows.

Later, in an important paper, Mauduit and Rivat [6] could handle the sum of digits of prime numbers.

Theorem 1.1 seems to be one of the first examples of a sequence with level of distribution equal to 1. More precisely, it is one of the first such examples where we have a maximum over a inside the sum over d; therefore the formulation "analogue of the Elliott–Halberstam conjecture".

Our second result concerns the Thue–Morse sequence on Piatetski-Shapiro sequences $\lfloor n^c \rfloor$.

Theorem 1.2. *Let $1 < c < 2$. The Thue–Morse sequence along $\lfloor n^c \rfloor$ is simply normal. That is, each of the letters* `0` *and* `1` *appears with asymptotic frequency $1/2$ in $n \mapsto \mathbf{t}(\lfloor n^c \rfloor)$.*

The two main theorems are connected. More precisely, by the same method of proof used for Theorem 1.1, we prove a Beatty sequence analogue; this analogue can be used to prove Theorem 1.2 using linear approximation of $\lfloor n^c \rfloor$ by $\lfloor n\alpha + \beta \rfloor$.

Theorem 1.3. *We define*

$$A(y, z; \alpha, \beta) = \left|\left\{y \le m < z : \mathbf{t}(m) = \mathtt{0} \text{ and } \exists n \in \mathbb{Z} \text{ such that } m = \lfloor n\alpha + \beta \rfloor\right\}\right|.$$

Let $0 < \theta_1 \le \theta_2 < 1$. There exist $\eta > 0$ and C such that

$$\int_D^{2D} \max_{\substack{y,z \\ 0 \le y \le z \\ z-y \le x}} \max_{\beta \ge 0} \left| A(y, z; \alpha, \beta) - \frac{z-y}{2\alpha} \right| \mathrm{d}\alpha \le C x^{1-\eta}$$

for all x and D such that $x \ge 1$ and $x^{\theta_1} \le D \le x^{\theta_2}$.

The method of approximation of $\lfloor n^c \rfloor$ by $\lfloor n\alpha + \beta \rfloor$ for proving theorems like Theorem 1.2 has been discussed in length in the earlier articles [8] by the author and [7] by Müllner and the author. In those articles, we obtained the ranges $1 < c \le 1.42$ and $1 < c < 1.5$ respectively, improving on the range $1 < c < 1.4$ obtained by Mauduit and Rivat [4].

Mauduit and Rivat [5], in another major paper, showed (in particular) that the Thue–Morse sequence along n^2 is simply normal.

Theorem 1.2 therefore closes the gap $[1.5, 2)$ in the set of exponents for which we have an asymptotic formula for Thue–Morse along $\lfloor n^c \rfloor$.

In the present article, we restrict ourselves to giving an outline of the proof of Theorem 1.1.

1.1. **Idea of proof of Theorem 1.1.** As in our earlier paper with Müllner [7, Section 4.1], it is sufficient to prove the following result.

Proposition 1.4. *For real numbers $N, D \ge 1$ and ξ set*

$$S_0 = S_0(N, D, \xi) = \sum_{D \le d < 2D} \max_{a \ge 0} \left| \sum_{0 \le n < N} \mathrm{e}\left(\frac{1}{2} s(nd + a)\right) \mathrm{e}(n\xi) \right|,$$

where $\mathrm{e}(x) = \exp(2\pi i x)$. Let $\rho_2 \ge \rho_1 > 0$. There exists an $\eta > 0$ and a constant C such that

$$\frac{S_0}{ND} \le C N^{-\eta}$$

holds for all $\xi \in \mathbb{R}$ and all real numbers $N, D \ge 1$ satisfying $N^{\rho_1} \le D \le N^{\rho_2}$.

It is sufficient to prove this for $D = 2^\nu$. By Cauchy–Schwarz, followed by van der Corput's inequality, moreover by using a "carry propagation lemma" (e.g. [7, Lemma 3.6]) we obtain

$$\begin{aligned}\left|S_0(N,2^\nu,\xi)\right|^2 \ll \frac{2^\nu N}{R_0} &\sum_{1\le r_0<R_0} \\ &\times \sum_{2^\nu\le d<2^{\nu+1}} \sup_{a\ge 0}\left|\sum_{0\le n<N} \mathrm{e}\left(\frac{1}{2}s_\lambda\big((n+r_0)d+a\big)-\frac{1}{2}s_\lambda\big(nd+a\big)\right)\right| + \text{errors},\end{aligned}$$

where s_λ is the "truncated sum-of-digits function" defined by $s_\lambda(n) = s(n \bmod 2^\lambda)$. We apply the inequalities of Cauchy–Schwarz and van der Corput alternatingly m times, and obtain

$$\left|\frac{S_0(N,2^\nu,\xi)}{2^\nu N}\right|^{2^{m+1}} \ll \frac{1}{R_0 2^{m\rho}2^\nu N} \sum_{\substack{1\le r_0<R_0\\ 0\le r_i<2^\rho, 1\le i\le m}} \sum_{0\le d<2^\lambda} \sup_{a\ge 0}\left|S_1\right| + \text{errors}, \tag{1.1}$$

where

$$S_1 = \sum_{0\le n<N} \mathrm{e}\left(\frac{1}{2}\sum_{\varepsilon_0,\ldots,\varepsilon_m\in\{0,1\}} s_\lambda\big(nd+a+\varepsilon_0 r_0 d+\varepsilon_1 r_1 K_1 d+\cdots+\varepsilon_m r_m K_m d\big)\right)$$

and $\lambda > \nu$ is chosen later in the proof.

Now we choose the multiples $K_1,\ldots,K_m$ in such a way that the number of digits to be taken into account is reduced from λ to $\rho := \lambda-(m+1)\mu$, where μ and m are chosen later. For this we use Farey series. Let $p_Q(\alpha)/q_Q(\alpha)$ be the element of the Farey series $\mathcal{F}_Q$ associated to α (see e.g. [7]) and set

$$\begin{aligned}K_1 &= 2^{2\mu} q_{2^\sigma}\left(\frac{d}{2^{(m-1)\mu}}\right);\\ K_i &= q_{2^{\mu+2\sigma}}\left(\frac{d}{2^{(i+1)\mu}}\right) q_{2^\sigma}\left(\frac{p_{2^{\mu+2\sigma}}\big(d/2^{(i+1)\mu}\big)}{2^{(m-i)\mu}}\right) \quad \text{for } 2\le i<m;\\ K_m &= q_{2^{\mu+\sigma}}\left(\frac{d}{2^{(m+1)\mu}}\right),\end{aligned}$$

where σ is chosen later. Using the approximation property $|q_Q(\alpha)\alpha - p_Q(\alpha)| < 1/Q$, we see that $K_i\alpha/2^{i\mu}$ is close to a multiple of 2^μ for $2 \le i \le m$. Using this, and the discrepancy of $n\alpha$-sequences modulo 1, we cut off μ many digits at a time (2μ many in the first step) and obtain

$$S_1 = S_2 + \text{errors},$$

where

$$S_2 = \sum_{0\le n<N} \mathrm{e}\left(\frac{1}{2}\sum_{\varepsilon_0,\ldots,\varepsilon_m\in\{0,1\}} s_{\lambda-(m+1)\mu}\left(\left\lfloor\frac{nd+a}{2^{(m+1)\mu}}+\frac{\varepsilon_0 r_0 d}{2^{(m+1)\mu}}\right\rfloor + \sum_{1\le i\le m}\varepsilon_i r_i \mathfrak{p}_i\right)\right),$$

and

$$\begin{aligned}\mathfrak{p}_1 &= p_{2^\sigma}\left(\frac{d}{2^{(m-1)\mu}}\right);\\ \mathfrak{p}_i &= p_{2^\sigma}\left(\frac{p_{2^{\mu+2\sigma}}\big(d/2^{(i+1)\mu}\big)}{2^{(m-i)\mu}}\right) \quad \text{for } 2\le i<m;\\ \mathfrak{p}_m &= p_{2^{\mu+\sigma}}\left(\frac{d}{2^{(m+1)\mu}}\right).\end{aligned} \tag{1.2}$$

For simplicity of argument, assume that the second summand in the floor function in the definition of S_2 is not present, and write $\widetilde{S}_2$ for the resulting expression. Using the usually very small discrepancy of $n\alpha$-sequences, and choosing m large enough, we see that the sequence $\lfloor (nd+a)/2^{(m+1)\mu} \rfloor$ runs through all the residue classes mod 2^ρ, where $\rho = \lambda - (m+1)\mu$, in a very uniform way. Therefore

$$\widetilde{S}_2 = \frac{N}{2^\rho} \sum_{0 \le k < 2^\rho} \mathrm{e}\left(\frac{1}{2} \sum_{\varepsilon_1,\ldots,\varepsilon_m \in \{0,1\}} s_\rho \left(k + \sum_{1 \le i \le m} \varepsilon_i r_i \mathfrak{p}_i \right) \right) + \text{errors}.$$

We note the important fact that this expression is independent of the residue class $a + d\mathbb{Z}$. This allows us to remove the maximum over a inside the sum over d, and thus prove the strong statement on the level of distribution.

Using the identity (1.1), we see that we have to treat the sum

$$\widetilde{S}_3 = \sum_{0 \le d < 2^\lambda} \sum_{0 \le r_1,\ldots,r_m < 2^\rho} \left| \widetilde{S}_2 \right|.$$

At this point, we apply the argument that for most $\alpha < 2^\lambda$ the 2-valuation of $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ is small. (This is established by a technical lemma that we do not present here.) For these α, the term $r_i \mathfrak{p}_i \bmod 2^\rho$ attains each $k \in \{0, \ldots, 2^\rho - 1\}$ not too often, as r_i runs. We may therefore replace $r_i \mathfrak{p}_1$ by r_i and thus obtain full sums over r_i. It remains to estimate the expression

$$\sum_{0 \le r_1,\ldots,r_m < 2^\rho} \left| \sum_{0 \le n < 2^\rho} \mathrm{e}\left(\frac{1}{2} \sum_{\varepsilon_1,\ldots,\varepsilon_m \in \{0,1\}} s_\rho \left(n + \sum_{1 \le i \le m} \varepsilon_i r_i \right) \right) \right|.$$

Using Cauchy–Schwarz, the absolute value can be removed at the cost of an additional variable r_{m+1}. The problem is therefore reduced to proving a nontrivial estimate for the sum

$$\sum_{0 \le r_1,\ldots,r_{m+1} < 2^\rho} \sum_{0 \le n < 2^\rho} \mathrm{e}\left(\frac{1}{2} \sum_{\varepsilon_1,\ldots,\varepsilon_{m+1} \in \{0,1\}} s_\rho \left(n + \sum_{1 \le i \le m+1} \varepsilon_i r_i \right) \right).$$

This is a *Gowers uniformity norm* for the Thue–Morse sequence; a very similar estimate was given by Konieczny [3], and we use ideas from that paper to prove our estimate. This finishes the proof.

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Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Wiedner Hauptstrasse 8–10, 1040 Vienna, Austria

Some negative results related to Poissonian pair correlation problems

Gerhard Larcher, Wolfgang Stockinger

Abstract

We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in $[0,1)$ has Poissonian pair correlations if

$$\lim_{N\to\infty} \frac{1}{N}\#\left\{1 \le l \ne m \le N : \|x_l - x_m\| \le \frac{s}{N}\right\} = 2s$$

for every $s \ge 0$. The aim of this talk is twofold. First, we will establish a gap theorem which allows to deduce that a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers in $[0,1)$ having a certain weak gap structure, cannot have Poissonian pair correlations. This result covers a broad class of sequences, e.g., Kronecker sequences, the van der Corput sequence and in more general LS-sequences of points and digital $(t,1)$-sequences. Additionally, this theorem enables us to derive negative pair correlation properties for sequences of the form $(\{a_n\alpha\})_{n\in\mathbb{N}}$, where $(a_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of integers with maximal order of additive energy, a notion that plays an important role in many fields, e.g., additive combinatorics, and is strongly connected to Poissonian pair correlation problems. These statements are not only metrical results, but hold for all possible choices of α.

Second, we study the pair correlation statistics for sequences of the form, $x_n = \{b^n\alpha\},\ n = 1,2,3,\ldots$, with an integer $b \ge 2$, where we choose α as the Stoneham number and as an infinite de Bruijn word.

1 Introduction and statement of main results

The concept of Poissonian pair correlations has its origin in quantum mechanics, where the spacings of energy levels of integrable systems were studied. See for example [1] and the references cited therein for detailed information on that topic. Rudnik and Sarnak first studied this concept from a purely mathematical point of view and over the years the topic has attracted wide attention, see e.g., [7, 14, 15, 16, 17].

Let $\|\cdot\|$ denote the distance to the nearest integer. A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers in $[0,1)$ has Poissonian pair correlations if the pair correlation statistics

$$F_N(s) := \frac{1}{N}\#\left\{1 \le l \ne m \le N : \|x_l - x_m\| \le \frac{s}{N}\right\} \tag{1}$$

tends to $2s$, for every $s \geq 0$, as $N \to \infty$.

Recently, Aistleitner, Larcher, Lewko and Bourgain (see [2]) could give a strong link between the concept of Poissonian pair correlations and the additive energy of a finite set of integers, a notion that plays an important role in many mathematical fields, e.g., in additive combinatorics. To be precise, for a finite set A of reals the additive energy $E(A)$ is defined as

$$E(A) := \sum_{a+b=c+d} 1,$$

where the sum is extended over all quadruples $(a, b, c, d) \in A^4$. Roughly speaking, it was proved in [2] that if the first N elements of an increasing sequence of distinct integers $(a_n)_{n\in\mathbb{N}}$, have an arbitrarily small energy saving, then $(\{a_n\alpha\})_{n\in\mathbb{N}}$ has Poissonian pair correlations for almost all α. In this paper the authors also raised the question if $(\{a_n\alpha\})_{n\in\mathbb{N}}$, where (a_n) is an increasing sequence of distinct integers with maximal order of additive energy, can have Poissonian pair correlations for almost all α. Jean Bourgain could show that the answer to this question is negative, i.e., he proved:

Theorem A (in [2]) *If $E(A_N) = \Omega(N^3)$, where A_N denotes the first N elements of $(a_n)_{n\in\mathbb{N}}$, then there exists a subset of $[0, 1]$ of positive measure such that for every α from this set the pair correlations of $(\{a_n\alpha\})_{n\in\mathbb{N}}$ are not Poissonian.*

Recently, the result of Bourgain has been further extended, see [1, 9, 10, 11]. The result given in [10] is an easy consequence of our Theorem 1 stated below and will be shown in Section 2. Further, see [4, 5] for (negative) results and discussions concerning a Khintchine type criterion which fully characterizes the metric pair correlation property in terms of the additive energy.

Due to a result by Grepstad and Larcher [6] (see also [3, 19]), we know that a sequence which satisfies that (1) tends to $2s$, for every $s \geq 0$, as $N \to \infty$, is also uniformly distributed in $[0, 1)$, i.e., it satisfies

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \leq n \leq N : x_n \in [a, b)\} = b - a$$

for all $0 \leq a < b \leq 1$. Note that the other direction is not necessarily correct. For instance the Kronecker sequence $(\{n\alpha\})_{n\in\mathbb{N}}$ is uniformly distributed modulo 1 for irrational α, but does not have Poissonian pair correlations for any real α; a fact that easily follows from continued fractions arguments. In earlier papers (see e.g., [2, 17]) this fact was argued to be an immediate consequence of the Three Gap Theorem [18]. The Three Gap Theorem, roughly speaking, states that the Kronecker sequence always has at most three distinct distances between nearest sequence elements. Nonetheless – at least for us – it is not immediately clear that we can deduce from this fact that $(\{n\alpha\})_{n\in\mathbb{N}}$ is not Poissonian for any α. Therefore, we will prove the following very general result concerning the link between Poissonian pair correlations and a certain gap structure of a sequence in the unit interval. In the next section, we will present some applications of this Theorem 1.

Theorem 1 *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence in* $[0,1)$ *with the following property: There is an* $s \in \mathbb{N}$*, positive real numbers* K *and* γ*, and infinitely many* N *such that the point set* $x_1, \ldots, x_N$ *has a subset with* $M \geq \gamma N$ *elements, denoted by* $x_{j_1}, \ldots, x_{j_M}$*, which are contained in a set of points with cardinality at most* KN *having at most* s *different distances between neighbouring sequence elements, so-called gaps. Then,* $(x_n)_{n\in\mathbb{N}}$ *does not have Poissonian pair correlations.*

Poissonian pair correlation is a typical property of a sequence. Random sequences, i.e., almost all sequences, have the Poissonian pair correlation property. Nevertheless, it seems to be extremely difficult to give explicit examples of sequences with Poissonian pair correlations. We note that $(\{\sqrt{n}\})_{n\in\mathbb{N}}$ has Poissonian pair correlations, [12] (see [13] for another explicit construction). Apart from that – to our best knowledge – no other explicit examples are known. Especially, until now we do not know any single explicit construction of a real number α such that the sequence of the form $(\{a_n\alpha\})_{n\in\mathbb{N}}$ has Poissonian pair correlations.

We recall that the sequence $(\{b^n\alpha\})_{n\in\mathbb{N}}$, for an integer $b \geq 2$, has the Poissonian property for almost all α. Moreover we know that the sequence $(\{b^n\alpha\})_{n\in\mathbb{N}}$ is uniformly distributed modulo 1 if and only if α is normal in base b, see e.g., [8]. So, if we want to investigate, whether the distribution of the pair correlations for some explicit given sequence is Poissonian, the sequence has to be uniformly distributed modulo 1. Therefore, if we study the distribution of the spacings between the sequence elements of $(\{b^n\alpha\})_{n\in\mathbb{N}}$, the only reasonable choice for α is a b-normal number. We will present two special instances, which were suggested by Yann Bugeaud as potential candidates in personal communication. We will consider so-called infinite de Bruijn words and Stoneham's number. For these instances, we also obtained negative results considering the Poissonian pair correlation structure.

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SYMMETRIC AND CONGRUENT RAUZY FRACTALS

PAUL SURER

Based on a joint work with K. Scheicher and V. Sirvent

Let $\mathcal{A} = \{1, 2, \ldots, m\}$ be a finite set (alphabet) and $\mathcal{A}^*$ the finite words over $\mathcal{A}$. We denote by σ a morphism $\mathcal{A}^* \longrightarrow \mathcal{A}^*$ that we require to be primitive, hence, there exists a power n such that for each two letters $i, j \in \mathcal{A}$ the word $\sigma^n(j)$ contains i at least once.

Primitivity ensures that the matrix $M_\sigma = (|\sigma(j)|_i)_{1\leq,i,j\leq m}$ (where $|\sigma(j)|_i$ denotes the number of occurrences of i in the word $\sigma(j)$) possesses a dominant real eigenvalue β. We are interested in unimodular Pisot substitutions, that is β is a Pisot unit of algebraic degree $d+1$ (thus, $d+1 \leq m$). In this case it is well known that we can associate to σ a compact set $\mathcal{R}_\sigma \subset \mathbb{R}^d$ of fractal shape known as Rauzy fractal (see, for example, [2, 3]).

Given unimodular Pisot substitutions σ, τ we are interested in the following problems:

- Which conditions ensure the associated Rauzy fractals $\mathcal{R}_\sigma$, $\mathcal{R}_\tau$ to be congruent (that is they differ by an affine transformation only)?
- Which conditions ensure that the Rauzy fractal $\mathcal{R}_\sigma$ is central symmetric?

We discuss the questions in terms of the induced language. For a primitive substitution σ it is defined by

$$\mathcal{L}_\sigma := \{A \in \mathcal{A}^* \,|\, \exists n \geq 1 : A \text{ is a subword of } \sigma^n(1)\}.$$

Therefore, our discourse involves a lot of combinatorics on words. Concretely we have the following two main results.

Theorem (*cf.* [5])**.** *Let σ and τ be irreducible Pisot substitutions (i.e. $d+1 = m$) over the same alphabet $\mathcal{A}$. If $\mathcal{L}_\sigma = \mathcal{L}_\tau$ then the Rauzy fractals $\mathcal{R}_\sigma$, $\mathcal{R}_\tau$ are congruent.*

For irreducible substitutions we need additional conditions.

For a word $A \in \mathcal{A}^*$ we denote by $\tilde{A}$ the reversed word (or *mirror-word*). We call a set $\mathcal{L} \subset \mathcal{A}^*$ *mirror-invariant* if for each $A \in \mathcal{L}$ we have $\tilde{A} \in \mathcal{L}$. With these notations we can state the following theorem concerning central symmetric Rauzy fractals.

Theorem (*cf.* [5])**.** *Let σ be a Pisot substitutions over $\mathcal{A}$. If $\mathcal{L}_\sigma$ is mirror-invariant then the Rauzy fractals $\mathcal{R}_\sigma$ is central-symmetric.*

Based on these two theorems we will present concrete classes of substitutions that yield congruent or central-symmetric Rauzy fractals. For these classes we will be able to give explicit expressions for the respective translation and the point

Key words and phrases. Substitutions; Rauzy fractals; Combinatorics on words.

of symmetry, respectively. Furthermore, we analyse whether the conditions in our theorems are necessary. We will see that the topic is related with up to now unsolved problems as the *Pisot conjecture* (see for example [1]) and the *Class $\mathcal{P}$ conjecture* stated in [4]. The presentation will be accompanied by illustrative examples.

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Institut für Mathematik, Universtiät für Bodenkultur (BOKU), Gregor-Mendel-Strasse 33, 1180 Wien, Austria

E-mail address: `paul@surer.at`

URL: `http://www.palovsky.com`

Digital questions in finite fields

Cathy Swaenepoel
Université d'Aix-Marseille

The connection between the arithmetic properties of an integer and the properties of its digits in a given basis produces a lot of interesting questions and many papers have been devoted to this topic. In the context of finite fields, the algebraic structure permits to formulate and study new problems of interest which might be out of reach in $\mathbb{N}$. This study was initiated by C. Dartyge and A. Sárközy.

We will devote our interest to several new questions in this spirit:

(1) estimate precisely the number of elements of some special sequences of $\mathbb{F}_q$ whose sum of digits is fixed;
(2) given subsets $\mathcal{C}$ and $\mathcal{D}$ of $\mathbb{F}_q$, find conditions on $|\mathcal{C}|$ and $|\mathcal{D}|$ to ensure that there exists $(c, d) \in \mathcal{C} \times \mathcal{D}$ such that the sum of digits of cd belongs to a predefined subset of $\mathbb{F}_p$;
(3) estimate the number of elements of an interesting sequence of $\mathbb{F}_q$ with preassigned digits.

We notice that the notion of digits in $\mathbb{F}_q$ is directly related to the notion of trace which is crucial in the study of finite fields.

SOME NEWS ON RATIONAL SELF-AFFINE TILES

J. M. THUSWALDNER

Let A be an expanding $d \times d$ integer matrix and $\mathcal{D} \subset \mathbb{Z}^d$ a complete set of residue class representatives of $\mathbb{Z}^d / A\mathbb{Z}^d$. Then by a classical result of Hutchinson one can uniquely define a nonempty compact set T by the set equation

$$AT = T + \mathcal{D}.$$

The set T is called a *self-affine tile*. Self-affine tiles have been studied extensively in the literature. An important result by Lagarias and Wang from 1997 states that such tiles form a lattice tiling w.r.t. the lattice $\mathbb{Z}^d$ under very general conditions. Together with Steiner we gave a generalization of this result to self-affine tiles defined in terms of rational matrices with irreducible characteristic polynomial. In the proofs of this generalization we used tools from algebraic number theory. In an ongoing work with Jankauskas and Steiner we want to get rid of the irreducibility condition. To this end we use methods from linear algebra and harmonic analysis.

Indecomposable integers and universal quadratic forms

Magdaléna Tinková (with M. Čech, D. Lachman, J. Svoboda and K. Zemková)

Department of Algebra, Charles University,
Sokolovská 83, 175 86 Praha 8, Czech Republic
tinkova.magdalena@gmail.com

This work focuses on the ring of algebraic integers $\mathcal{O}_K$ of totally real fields K and the semigroup $\mathcal{O}_K^+$ of totally positive integers. In this structure, we can define so-called indecomposable integers, elements of $\mathcal{O}_K^+$ which cannot be expressed as a sum of two totally positive integers. It is also not hard to see that each element of $\mathcal{O}_K^+$ can be written as a sum of finitely many indecomposables. In connection to number systems, we can express totally positive numbers using the indecomposable integers as a base.

In quadratic fields $\mathbb{Q}(\sqrt{D})$, there were characterized all indecomposable integers using the continued fraction of $\sqrt{D}$ or $\frac{\sqrt{D}-1}{2}$, see [14, 5]. In the second article, the authors also found the upper bound on the norm of indecomposables, which was refined in [7]. We do not have such a characterization for fields of higher degrees. However, in [3], there was derived a bound on norms. This bound can be greatly improved in the quadratic case [5, 10] and we will discuss some original results related to these estimates.

The indecomposable integers are closely related to quadratic forms. By this, we mean

$$Q(x_1,x_2,\ldots,x_n)=\sum_{1\leq i\leq j\leq n} a_{ij}x_ix_j$$

where $a_{ij}\in\mathcal{O}_K$ and $x_1,x_2,\ldots,x_n$ are variables. We focus on totally positive definite forms, i.e., $\mathbb{Q}(\gamma_1,\ldots,\gamma_n)$ is totally positive for all $\gamma_i\in\mathcal{O}_K$ such that our n-tuples are not equal to zero.

Our form is called universal if it represents each element of $\mathcal{O}_K^+$. For example, $x_1^2+x_2^2+x_3^2+x_4^2$ is universal over $\mathbb{Z}$. Moreveover, in [16], there was proved that the sum of any number of squares is universal only over $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{5})$. Other results we can see in [4, 2, 12]. The indecomposable integers play an important role in the study of universal quadratic forms, since they are difficult to represent and we often use them as coefficients of our forms.

In biquadratic fields $\mathbb{Q}(\sqrt{p},\sqrt{q})$, we focus on the question whether the indecomposable integers from quadratic subfields remain indecomposable in $\mathbb{Q}(\sqrt{p},\sqrt{q})$. The following theorem shows one of our results, which can be improved under certain conditions.

Theorem 1. *Let* $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ *be a biquadratic field, and let* $r = \frac{pq}{\gcd(p,q)^2}$. *Set*

$$\delta_p = \begin{cases} \sqrt{p} & \text{if } p \equiv 2,3 \ (mod\ 4), \\ \frac{\sqrt{p}-1}{2} & \text{if } p \equiv 1 \ (mod\ 4). \end{cases}$$

If

$$\sqrt{r} > \sqrt{p} \max\{u_i;\ i \text{ odd, } [u_0, \overline{u_1, u_2, \ldots, u_{s-1}, u_s}] \text{ is the continued fraction of } \delta_p\},$$

then the indecomposable integers from $\mathbb{Q}(\sqrt{p})$ *are indecomposable in* $\mathbb{Q}(\sqrt{p}, \sqrt{q})$.

We will also discuss how we can use our knowledge of indecomposable integers in biquadratic fields for the study of universal forms, in particular, we can show that each universal form over $\mathbb{Q}(\sqrt{6}, \sqrt{19})$ must have at least 6 variables.

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Lenticular Poles of the Dynamical Zeta Function of the β-shift for Simple Parry Numbers Close to One

Denys Dutykh and *Jean-Louis Verger-Gaugry**

Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA,
F-73000 Chambéry, France

* *Corresponding author's email:* `Jean-Louis.Verger-Gaugry@univ-smb.fr`

Abstract. The distribution of the conjugates and the $\beta-$conjugates of simple PARRY's numbers in SOLOMYAK's fractal is revisited in the case where the base of numeration $\beta > 1$ is close to one. We focus on the class $\mathscr{B}$ of polynomials of the type

$$f(x) = -1 + x + x^{n} + x^{m_1} + \ldots + x^{m_s}, \qquad s \geqslant 1, \qquad n \geqslant 2,$$

with $m_1 - n \geqslant n-1$, $m_{q+1} - m_q \geqslant n-1$, $q = 1, 2, \ldots, s-1$. They are polynomials sections of PARRY Upper functions associated with the dynamical zeta functions of the $\beta-$shift, in the RÉNYI-PARRY dynamical system in base β. This work has two objectives: (*i*) to show that there exists an asymptotic "*reducibility Conjecture*" for the polynomials of the class $\mathscr{B}$ in the context of ODLYZKO–POONEN's Conjecture, which provides an asymptotics on the type of $\beta-$conjugates, this reducibility conjecture is empirically supported by extensive MONTE-CARLO simulations, (*ii*) to establish new theorems of factorization of the polynomials of the class $\mathscr{B}$, by comparison with the general theorems of SCHINZEL related to the reducibility of lacunary polynomials. In particular, in the search of reciprocal integer polynomials having a small MAHLER measure by the PARRY Upper function we will show the existence of lenticuli of roots, described by divergent series (asymptotic expansions à la POINCARÉ), in the region of the cusp of SOLOMYAK's fractal and indicate how they are related to the problem of LEHMER.

AN EFFECTIVE CRITERION FOR PERIODICITY OF ℓ-ADIC CONTINUED FRACTIONS

FRANCESCO VENEZIANO
(JOINT WORK WITH L. CAPUANO AND U. ZANNIER)

Extended abstract

The theory of real continued fractions plays a central role in real Diophantine Approximation for many different reasons, in particular because the convergents of the simple continued fraction expansion of a real number α give the best rational approximations to α. Motivated by the same type of questions, several authors (see for example Mahler [Mah34], Schneider [Sch70], Ruban [Rub70], Bundschuch [Bun77], and Browkin [Bro78]) have generalized the theory of real continued fractions to the ℓ-adic case in various ways.

There is not a canonical way to define a continued fraction expansion in this context, as we lack a canonical ℓ-adic analog of the integral part. The ℓ-adic process which is the most similar to the classical real one was mentioned for the first time in one of the earliest papers on the subject by Mahler [Mah34], and then studied accurately by Ruban [Rub70], who showed that these continued fractions enjoy nice ergodic properties.

Ruban's continued fractions will be the subject of this talk and they have many important differences with respect to the classical real ones. First of all, while some rational numbers have a finite expansion, this is not — unlike the real case — the only possible behaviour. For example, it is easy to see that negative rational numbers cannot admit a terminating Ruban continued fraction.

It is however possible to decide when a given rational number admits a finite Ruban continued fraction expansion and indeed our first result is the following:

Theorem 1. *Let ℓ be a prime number and $\alpha \in \mathbb{Q}$ be a rational number.*

(i) *The Ruban continued fraction expansion of α terminates if and only if all complete quotients are non-negative; there is an algorithm to decide in a finite number of steps whether this happens.*

(ii) *If the Ruban continued fraction expansion of α does not terminate, then it is periodic with all partial quotients eventually equal to $\ell - \ell^{-1}$; in this case, the pre-periodic part can be effectively computed.*

Disregarding the computability aspect, the last part of this result has already appeared in the literature, due to Laohakosol and, independently, to Wang (see [Lao85] and [Wan85]), but this does not seem to be the case for either of the algorithmic conclusions, which apparently do not follow directly from the proofs in [Lao85] and [Wan85]. For completeness, we have also included our own (short) proof of the qualitative part, which is quite different.

The conclusion of Theorem 1 depends of course on the precise algorithm defining the continued fraction expansion. In [Bro78], Browkin modified Ruban's definition so that every rational number has a finite ℓ-adic continued fraction expansion.

Another natural question arises when one considers the periodicity of Ruban continued fractions. In the classical real case, Lagrange's theorem states that a real number has an infinite periodic continued fraction if and only if it is quadratic irrational. We will show that this is not true in the ℓ-adic case and only some similarities can be recovered. For example we prove the following result:

Theorem 2. *A Ruban continued fraction which is periodic represents an element of $\mathbb{Q}_\ell$ which is either a rational number or a real quadratic irrational over $\mathbb{Q}$.*

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No full analogue of Lagrange's theorem holds in this setting as remarked by Ooto in [Oot17], and the problem of deciding whether a quadratic (ℓ-adic) irrational number has a periodic continued fraction seems still open. For Browkin's definition, some very partial sufficient conditions were given in a series of papers by Bedocchi [Bed93]. Moreover, in [Bro01], Browkin wrote an algorithm to generate the periodic continued fraction expansion of $\sqrt{\Delta} \in \mathbb{Q}_\ell \setminus \mathbb{Q}$ for some values of Δ and ℓ, giving many numerical examples.

In this paper we investigate the periodicity of the ℓ-adic Ruban continued fraction expansion of quadratic irrational numbers, thus solving a problem posed by Laohakosol in [Lao85].

Our main result is the following:

Theorem 3. *Let $\alpha \in \mathbb{Q}_\ell \setminus \mathbb{Q}$ be a quadratic irrational over $\mathbb{Q}$. Then, the Ruban continued fraction expansion of α is periodic if and only if there exists a unique real embedding $j : \mathbb{Q}(\alpha) \to \mathbb{R}$ such that the image of each complete quotient α_n under the map j is positive.*

Moreover, there is an effectively computable constant N_α such that, either $\exists\, n \leq N_\alpha$ such that both real embeddings of α_n are negative, and therefore the expansion is not periodic, or $\exists\, n_1 < n_2 \leq N_\alpha$ such that $\alpha_{n_1} = \alpha_{n_2}$, hence the expansion is periodic.

In particular, both the preperiodic and the periodic part of a periodic expansion can be computed with a finite algorithm.

I summarise here a simplified version of the explicit bounds that we prove.

- If the expansion of a rational α is finite, then its length is at most $\frac{\log H(\alpha)}{\log \ell} + 2$;
- If the expansion of a rational α is periodic, then the length of the preperiodic part is at most $32\ell H(\alpha)^2$;
- If $\alpha = \frac{b+\sqrt{\Delta}}{c}$, with b, c, Δ integers, $\Delta > 0$ not a square, then the constant N_α in Theorem 3 can be bounded by $bc + 2(c\sqrt{\Delta} + 1)^3$.

It is also interesting to study how the qualitative behaviour of the expansion varies with the prime ℓ for fixed rational or irrational quadratic numbers. We show that finiteness of the expansion (for rational numbers) and periodicity (for irrational quadratic numbers) are "unlikely" behaviours which occur for at most finitely many primes.

This extended abstract has been adapted from the introduction of the full paper, which is available on ArXiv at the address `https://arxiv.org/abs/1801.06214`.

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Collegio Puteano, Scuola Normale Superiore, Piazza dei Cavalieri, 3, I-56100 PISA, Italia

E-mail address: `francesco.veneziano@sns.it`

PERIODIC REPRESENTATIONS IN SALEM BASES

TOMÁŠ VÁVRA

This contribution is a continuation of author's work on periodic representations in number systems with algebraic bases. By the results of Baker, Kala, Masáková, Pelantová, and Vávra, we know that for an algebraic base β without Galois conjugates on the unit circle there is a finite alphabet $\mathcal{A} \subset \mathbb{Z}$ such that each element of $\mathbb{Q}(\beta)$ admits an eventually periodic $(\beta, \mathcal{A})$-representation. That is, a representation of the form $\sum_{i=-k}^{+\infty} a_i \beta^{-i},\ a_i \in \mathcal{A}$ with the digit sequence $\{a_i\}$ being eventually periodic.

We will show that the mentioned result can be strengthened to cover all the algebraic bases, in particular also the Salem numbers, for which this problem was unapproachable by the previous method. Moreover, we discuss whether the representations constructed by our method satisfy the weak-greedy condition, i.e., whether the leading power of the representation is proportional to the modulus of the represented number. We come to the conclusion that the weak-greedy condition is satisfied if and only if $|\beta|$ is a Pisot or a Salem number, or β is a complex Pisot or a complex Salem number.

Furthemore, we study the decideability whether a pair $(\beta, \mathcal{A})$ satisfies the periodicity condition. We show that the periodicity condition can be equivalently stated as a topological property of the attractor of certain iterated function system, or as a geometric property of the spectrum of β with the alphabet $\mathcal{A}$.

Department of Algebra, Charles University, Sokolovská 83, 175 86 Praha 8, Czech Republic

TOPOLOGY OF A CLASS OF SELF-AFFINE TILES

JÖRG THUSWALDNER AND SHU-QIN ZHANG (SPEAKER)

ABSTRACT. Let $T = T(M, \mathcal{D})$ be an integral self-affine tile ($\mathbb{Z}^3$-tile) in $\mathbb{R}^3$ generated by an expanding matrix

$$M = \begin{pmatrix} 0 & 0 & -C \\ 1 & 0 & -B \\ 0 & 1 & -A \end{pmatrix} \text{ and the digit set } \mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} C-1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

where $1 \leq A \leq B < C$. To study the topological properties of a $\mathbb{Z}^3$-tile, the neighbor set plays an important role. We have the following result about neighbor set. For $1 \leq A < B < C$, the $\mathbb{Z}^3$-tile has 14-neighbors if and only if one of the following conditions satisfies.

(1). $B \geq 2A - 1$ and $C \geq 2(B - A) + 2$;

(2). $B < 2A - 1$ and $C \geq A + B - 2$.

In particular, if $A = B = 1$, it has 20 neighbors for all $C \geq 2$. Moreover, we will also show when the boundaries of the self-affine tiles are topological 2-spheres.

CHAIR OF MATHEMATICS AND STATISTICS, UNIVERSITY OF LEOBEN, FRANZ-JOSEF-STRASSE 18, A-8700 LEOBEN, AUSTRIA

E-mail address: joerg.thuswaldner@unileoben.ac.at

E-mail address: zhangsq_ccnu@sina.com

Author Index