On the Spectra of Numbers

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Outline

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From 1990 P. Erdős, I. Joó, V. Komornik and others

Definition

Given $\beta > 1$ and $m \in \mathbb{N}$.

$$X^m(\beta) = \left\{ \varepsilon_0 + \varepsilon_1\beta + \ldots + \varepsilon_n\beta^n \mid n \in \mathbb{N}, \varepsilon_i \in \{0, 1, 2, \ldots, m\} \right\}$$

is the spectrum of β.

P. Borwein, K. Hare: *Some computations on the spectra of Pisot and Salem numbers*, Math. Comp. (2002)

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$Y^m(\beta) = X^m(\beta) - X^m(\beta)$ is a Meyer set having a scaling invariance. This gives a number theoretical contruction of Meyer sets which play an important role in quasi-crystals analysis."

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is the spectrum of β.

In $[0, \beta^N)$, only finitely many points of $X^m(\beta) \implies \exists (x_n)$

$$0 = x_0 < x_1 < x_2 < \ldots \qquad \text{and} \qquad \{x_n \mid n \in \mathbb{N}\} = X^m(\beta)$$

Task: Describe properties of $X^m(\beta)$, in particular

$$\ell^m(\beta) = \liminf_{n\to\infty}(x_{n+1} - x_n) \qquad \text{and} \qquad L^m(\beta) = \limsup_{n\to\infty}(x_{n+1} - x_n)$$

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Simple properties of $X^m(\beta)$

$$X^m(\beta, N) := \left\{ \sum_{i=0}^{N} a_i \beta^i \ : , a_i \in \{0 \ldots, m\} \right\}, \quad N \in \mathbb{N} \text{ is fixed}$$

Then

$$X^m(\beta, N+1) = \beta X^m(\beta, N) + \{0, \ldots, m\}$$

By induction on N:

- If $m \geq \beta - 1$, then distances between neighbours in $X^m(\beta, N)$ are at most 1.

$$\ell^m(\beta) \leq L^m(\beta) \leq 1.$$

- If $m < \beta - 1$, then distances between neighbours in $X^m(\beta, N)$ are at least 1. Moreover, the distance of the neighbours $z_1 = \sum_{k=0}^{N-1} m\beta^k = m\frac{\beta^N - 1}{\beta - 1}$ and $z_2 := \beta^N$ is $\frac{\beta^N(\beta - 1 - m) + m}{\beta - 1}$.

$$1 = \ell^m(\beta) < L^m(\beta) = +\infty.$$

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When $\ell^m(\beta) > 0$?

A Pisot number is an algebraic integer > 1 all of whose conjugates have modulus < 1.

Theorem

If β is Pisot, then $\ell^m(\beta) > 0$ for any $m \in \mathbb{N}$.

Proof: (A.M. Garsia 1962)
Let $\beta = \beta_1$ be of degree d and $\beta_2, \beta_3, \ldots, \beta_d$ its conjugates.

$\sigma_i : \mathbb{Q}(\beta) \mapsto \mathbb{Q}(\beta_i)$ - the field isomorphism

$y = x_{n+1} - x_n \in \left\{ \sum_{i=0}^{s} b_i \beta^i \ : \ s \in \mathbb{N}, b_i \in \{-m, \ldots, m\} \right\}$

$y = \sum_{k=0}^{s} b_k \beta^k \implies |\sigma_i(y)| \leq \sum_{k=0}^{\infty} m|\beta_i|^k = \frac{m}{1-|\beta_i|}$ for $i \geq 2$

$$1 \leq |N(y)| := \prod_{i=1}^{d} |\sigma_i(y)| \leq y \prod_{i=2}^{d} \frac{m}{1-|\beta_i|}$$

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$\ell^m(\beta) = 0$ if and only if 0 is an accumulation point of the set

$$Y^m(\beta) = X^m(\beta) - X^m(\beta)$$

Clearly, $Y^m(\beta) \subset Y^{m'}(\beta)$ for any $m < m'$. Thus the crucial question:

Is 0 an accumulation point of $Y^m(\beta)$ for $m = \lfloor \beta \rfloor$?

Theorem (Feng, 2016)

$Y^m(\beta)$ is dense in $\mathbb{R}$ if and only if $\beta > 1$ is not Pisot and $m > \beta - 1$.

D.-J. Feng, *On the topology of polynomials with bounded integer coeffcients*, J. of EMS (2016)

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Results on the topology of $Y^m(\beta)$

- V. Drobot, *On sums of powers of a number*, Amer. Math. Monthly (1973)
- V. Drobot, S. McDonalds, *Approximation properties of polynomials with bounded integers coefficients*, Pacific J. Math. (1980)
- Y. Bugeaud, *On a property of Pisot numbers and related questions*, Acta Math. Hungar. (1996)
- P. Erdős, V. Komornik, *On developments in non-integer bases*, Acta Math. Hungar. (1998)
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- N. Sidorov, B. Solomyak, *On the topology of sums in powers of an algebraic number*, Acta Arith. (2011)
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Geometry of $X^m(\beta)$ for Pisot bases

Corollary

Let β be a Pisot number and $m > \beta - 1$. The set of distances between neighbours of $X^m(\beta)$ is finite.

Thus $X^m(\beta)$ can be coded by an infinite word $\mathbf{u}^m(\beta)$ over a finite alphabet $\mathcal{A}$.

Example: $\beta = \frac{1+\sqrt{5}}{2}$ and $m = 1$. There are two distances in $X^m(\beta)$, namely $a = 1$ and $b = 1/\beta$. The set $X^m(\beta)$ is coded by the Fibonacci word, i.e. by the fixed point of the morphism $a \mapsto ab$ and $b \mapsto a$.

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Let β be a Pisot number and $m > \beta - 1$. Then the infinite word $\mathbf{u}^m(\beta)$ is a morphic image of a fixed point of a non-identical morphism.

The proof is constructive.

BUT: size of the alphabet is exaggerated and the morphism is not primitive

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Precise values of $\ell^m(\beta)$

Erdős, Joó (1992) $\beta^d = \beta^{d-1} + \ldots + \beta + 1$, $\ell^1(\beta) = \frac{1}{\beta}$

Komornik, Loreti, Pedicini (2000) $\beta^3 = \beta^2 + 1$, $\ell^1(\beta) = \beta^2 - 1$

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Komatsu(2002), Borwein & Hare (2003) $\ell^m(\beta)$
for any quadratic unit $\beta > 1$ and any $m > \beta - 1$

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Description of all gaps in $X^m(\beta)$ and their densities

Bugeaud (2002) $\beta^d = \beta^{d-1} + \ldots + \beta + 1$ and $m = 1$

Feng, Wen (2002) $\beta^3 - \beta^2 - 1 = 0$ and $1 \leq m \leq 10$

Feng, Wen (2002) for 10 smallest Pisot numbers and $m = 1$

Garth, Hare (2006) $\beta^2 = a\beta - b$, where $b + 2 \leq a$ and $m = \lfloor \beta \rfloor$

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β-integers

Č. Burdík, Ch. Frougny, J.-P.Gazeau, R. Krejcar, *Beta-integers as natural counting systems for quasicrystals* (1998)

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Bounded distance to a lattice

Definition

A discrete set $\Lambda \subset \mathbb{R}^d$ is bounded distance equivalent to a lattice $L \in \mathbb{R}^d$, if there exist a bijection $g : \Lambda \mapsto L$ and a constant $K > 0$ such that

$$||x - g(x)|| < K \quad \text{for all } x \in \Lambda.$$

- Let $\beta > 1$ be zero of $x^d - ax^{d-1} - ax^{d-2} - \ldots - ax - b$, where $b \le a$. The spectrum $X^m(\beta)$ with $m = \lfloor \beta \rfloor$ is bounded distance equivalent to a lattice $c\mathbb{Z}$.

Theorem (Fabre, 1995)

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Is $X^m(\beta)$ with $m > \lfloor\beta\rfloor$ BDE to a lattice?

β quadratic Pisot unit, $\mathcal{A} = \{0, \dots, m\} - s \subset \mathbb{Z}$, $0 < s < m$.

$$X^{\mathcal{A}}(\beta) = \Big\{ \sum_{k=0}^{n} a_k \beta^k \ : \ n \in \mathbb{N}, a_0, a_1, \dots, a_n \in \mathcal{A} \Big\}$$

Theorem (Masáková, Pastirčáková, P. , 2014)

There exist $\Delta_1, \Delta_2 > 0$ such that the gaps in $X^{\mathcal{A}}(\beta)$ take values in $\{\Delta_1, \Delta_2, \Delta_1 + \Delta_2\}$. Moreover, the bidirectional sequence of gaps in $X^{\mathcal{A}}(\beta)$ is a coding of exchange of three intervals.

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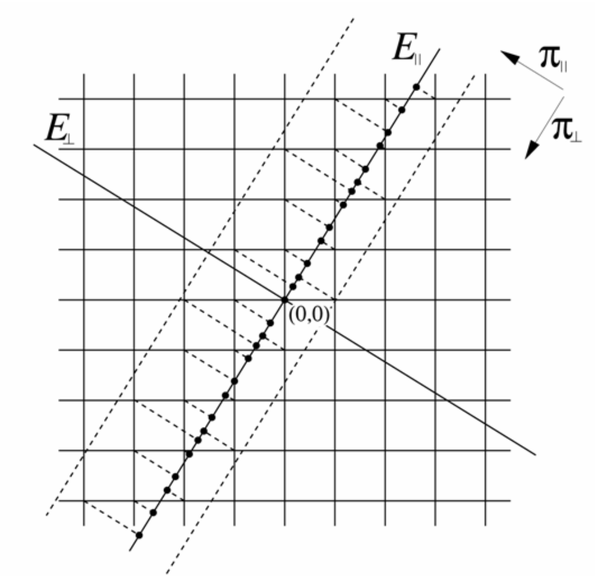
Spectrum as a Cut-and-project set



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On-line division and the spectrum

On-line division (Trivedi, Ercegovac 1978) requires:

1) numeration system $(\beta, \mathcal{A})$ is redundant
2) divisor in the form $y = 0 \bullet y_1 y_2 y_3 \cdots$ and $|y| > D_{min}$.

Definition

Let $z_1 z_2 \cdots$ be a $(\beta, \mathcal{A})$-representation of 0, i.e. $\sum_{i \geq 1} z_i \beta^{-i} = 0$. It is said to be *rigid* if

$$0 \bullet z_1 z_2 \cdots z_j \neq 0 \bullet 0 z_2' \cdots z_j'$$

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Example: $\beta = 2$ with alphabet $\{\bar{1}, 0, 1\}$.

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Preprocessing in $(\beta, \mathcal{A})$-numeration system is possible if and only if no $(\beta, \mathcal{A})$-representation of zero is rigid.

Theorem (Frougny, P., 2018)

Let β be a complex number, $|\beta| > 1$, and let $\mathcal{A} \subset \mathbb{C}$ be a finite alphabet. 0 has a rigid $(\beta, \mathcal{A})$-representation if and only if the spectrum $X^{\mathcal{A}}(\beta)$ has an accumulation point.

By Akiyama, Komornik 2013, and Feng 2016.

Corollary

In numeration system $(\beta, \mathcal{A})$, where

$$\mathbb{N} \ni m > \beta - 1 > 0 \quad \text{and} \quad \mathcal{A} = \{-m, \ldots, -1, 0, 1 \ldots, m\}$$

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Outline

Spectra in the complex plane

base $\beta \in \mathbb{C}$ of modulus > 1
alphabet $\mathcal{A} \subset \mathbb{C}$ – finite

$(\beta, \mathcal{A})$-spectrum:

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$$\ell^m(\beta) = \liminf_{n\to\infty}(x_{n+1} - x_n) \qquad \longrightarrow$$

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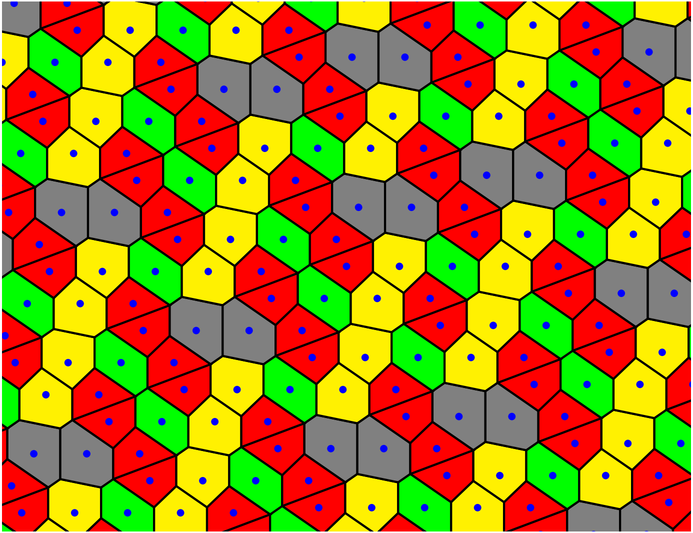

$X^m(\beta)$ and its Voronoi tiling
β - the complex zero of $x^3 + x^2 + x - 1$ and $m = 2$

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A complex Pisot number is a non-real algebraic integer > 1 in modulus whose Galois conjugates except its complex conjugate are < 1 in modulus.

Example: $\delta > 1$ a real Pisot number. Then $\beta = i\sqrt{\delta}$ is a complex Pisot number.

Theorem (Zaimi 2004)

Let β be a non-real number of modulus > 1. Then, β is a complex Pisot number if and only if $\ell_m(\beta) > 0$ for all m.

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If $m \geq \beta\overline{\beta} - 1$ and $\frac{1}{\gamma}$ has F-property then $L_m(\beta) < +\infty$

F-property for cubic Pisot units characterized:
— by Akiyama (1998), if $\gamma > 0$
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Y. Herreros (PhD thesis, 1991)

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Alphabets closed under multiplication

Theorem (Hare, Masáková, Vávra, 2017)

Let $\beta \in \mathbb{C}, |\beta| > 1$ and $\mathcal{A} \subset \mathbb{C}$ be finite. If $X^{\mathcal{A}}(\beta)$ is discrete, then

$$X^{\mathcal{A}}(\beta) \text{ is relatively dense} \iff \mathbb{C} \text{ is } (\beta, \mathcal{A})\text{-representable}.$$

Y. Herreros (PhD thesis, 1991)

$$\Delta_m = \{0\} \cup \{\omega^k : k = 1, 2, \ldots, m\} \text{ with } \omega = \exp(i\tfrac{2\pi}{m})$$

Herreros characterized $\beta > 1$ for which $\mathbb{C}$ is (β, Δ_m)-representable.

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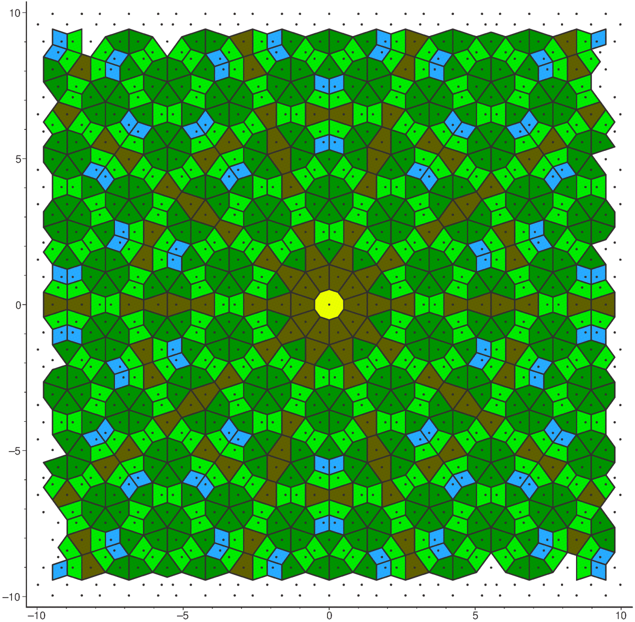
10
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Open problems

real Pisot number β

- Describe combinatorial properties of the word coding gaps in $X^m(\beta)$: factor complexity and balancedness.
- Characterize pairs (β, m) such that $X^m(\beta)$ is bounded distance equivalent to a lattice.

complex Pisot number β

- An analogy for Feng's theorem for $\beta \in \mathbb{C} \setminus \mathbb{R}$?

  $\ell_m(\beta) = 0$ if and only if $m > \beta\overline{\beta} - 1$ and β is not a complex Pisot.

- For a finite alphabet $\mathcal{A} \subset \mathbb{C}$, decide whether $\mathbb{C}$ is $(\beta, \mathcal{A})$-representable. An analogy of Pedicini's characterization we have for real bases is missing.

complex Pisot number β

- When $X^m(\beta)$ is bounded distance equivalent to a lattice?

  Example: $\beta = i\sqrt{\delta}$, δ real Pisot and $X^m(-\delta)$ is BDE to a lattice, then $X^m(\beta)$ BDE to a lattice.

Thank you !