Fundamental Domains for Rational Function Based Digit Systems of Formal Laurent Series over Finite Fields

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Outline

1. Number systems and tilings over Laurent series
   - Digit systems in $\mathbb{F}((x^{-1}, y^{-1}))/f\mathbb{F}((x^{-1}, y^{-1}))$
   - Fundamental Domains
2. Rational function based digit systems over finite fields
   - P/Q-digit systems in $\mathbb{F}[x]$
   - P/Q-digit systems in $\mathbb{F}((x^{-1}))$
3. Connection
   - Connecting P/Q-digit systems to tilings over Laurent series
   - Fundamental Domains for P/Q-digit systems
4. Recommendations for future work

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Digit systems in $\mathbb{F}((x^{-1},y^{-1}))/f\mathbb{F}((x^{-1},y^{-1}))$

- Given:
  - $\mathbb{F}$ a field
  - $f \in \mathbb{F}[x,y]$ monic in both x and y
  - $s \in S := \mathbb{F}((x^{-1},y^{-1}))/f\mathbb{F}((x^{-1},y^{-1}))$
- *x-digit representation of s* : $h = \sum_{i=-\infty}^{k} d_i x^i \in \mathbb{F}((x^{-1},y^{-1}))$ such that $s = h + f\mathbb{F}((x^{-1},y^{-1}))$, $k \in \mathbb{Z}$ and $d_i \in \mathbb{F}[y]$ with $\deg_y(d_i) < \deg_y(f)$

Theorem (Beck-Brunotte-Scheicher-Thuswaldner, 2009)

Let $f \in \mathbb{F}[x,y]$ be monic in both x and y. Then every element of S admits a unique x-digit representation.

- *x-digit string of s* : $\langle\langle s \rangle\rangle_x = d_k d_{k-1} \cdots d_0 \bullet d_{-1} \cdots$

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Example of x-digit representation

Example

Given:

- $\mathbb{F} = \mathbb{F}_5$
- $f = y^3 + (2x+1)y + 3x + x^2 \in \mathbb{F}[x,y]$
- $s = y^5 + 2x^4y^4 + f\mathbb{F}((x^{-1},y^{-1}))$

We get:

$$y^5 + 2x^4y^4 \equiv (3y)x^6 + (y^2+4y)x^5 + (3y^2)x^4 + (2)x^3 + (4y^2+4y+2)x^2 + (2y^2+4y+3)x + y \pmod{f}$$

$$\langle\langle s\rangle\rangle_x = 3y, y^2+4y, 3y^2, 2, 4y^2+4y+2, 2y^2+4y+3, y\bullet\overline{0}$$

Digit systems in $\mathbb{F}((x^{-1}, y^{-1}))/f\mathbb{F}((x^{-1}, y^{-1}))$

- Given: $s \in S$ with x-digit representation $h \in \mathbb{F}((x^{-1}, y^{-1}))$
- *x-height of s* : $\mathrm{hgt}_x(s) = \deg_x(h)$
- *x-norm* : $|s|_x := q^{\mathrm{hgt}_x(s)}$ where $q = \begin{cases} \#\mathbb{F} & \text{if } \mathbb{F} \text{ is finite} \\ e & \text{if } \mathbb{F} \text{ is infinite} \end{cases}$
- S is a complete topological $\mathbb{F}((x^{-1}))$-vector space wrt $|\cdot|_x$
- *x-fundamental domain of S* : $\mathcal{F}_x := \{s \in S \mid |s|_x < 1\}$

Mutual composition of fundamental domains

Proposition (Beck-Brunotte-Scheicher-Thuswaldner, 2009)

Set $m = \deg_x(f)$, $n = \deg_y(f)$, $\rho = (m-1)(n-1)$, *and*

$$V_x := \left\{ s \in \mathcal{F}_x \;\middle|\; s = \sum_{i=0}^{m-1} \sum_{j=-\rho}^{n-1} a_{ij} x^i y^j + f\mathbb{F}((x^{-1}, y^{-1})) \text{ for some } a_{ij} \in \mathbb{F} \right\}.$$

Then V_x *is an* $\mathbb{F}$*-vector space and*

$$\mathcal{F}_x = \bigcup_{s \in V_x} (s + y^{-\rho} \mathcal{F}_y).$$

In particular, $\mathcal{F}_x$ *is a clopen, bounded, and compact subset of the topological* $\mathbb{F}((y^{-1}))$*-vector space* S.

Tilings induced by fundamental domains

Theorem (Beck-Brunotte-Scheicher-Thuswaldner, 2009)

The collection $\{r + \mathcal{F}_x \mid r \in \mathbb{F}[x,y]/f\mathbb{F}[x,y]\}$ forms a tiling of the $\mathbb{F}((y^{-1}))$-space S in the sense that

$$S = \bigcup_{r \in \mathbb{F}[x,y]/f\mathbb{F}[x,y]} (r + \mathcal{F}_x)$$

and $(r + F_x) \cap (r' + F_x) = \varnothing$ for any distinct $r, r' \in \mathbb{F}[x,y]/f\mathbb{F}[x,y]$.

Visualization of fundamental domains when $\mathbb{F}$ is finite

Suppose $\mathbb{F} = \mathbb{F}_q$ is finite of order q. Let $u : \mathbb{F} \longrightarrow \{0, \ldots, q-1\}$ be an enumeration of $\mathbb{F}$. Consider the map $\underbrace{\mu \times \cdots \times \mu}_{m \text{ times}} : \mathbb{F}((y^{-1}))^m \longrightarrow \mathbb{R}^m$

where μ is defined by

$$\begin{aligned} \mu : \mathbb{F}((y^{-1})) &\longrightarrow \mathbb{R} \\ \sum_{i=-\infty}^{l} a_i y^i &\longmapsto \sum_{i=-\infty}^{l} u(a_i) q^i. \end{aligned}$$

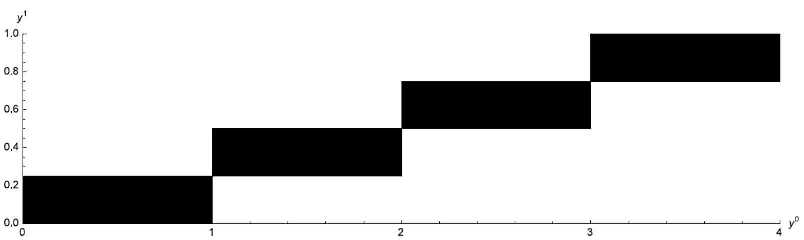
Visualization of fundamental domains when $\mathbb{F}$ is finite

Example

$f = x^3 + (y+1)x^2 + x + y^2 + y \in \mathbb{F}_2[x, y]$

$\implies n = 2$, V_y has an $\mathbb{F}_2$-basis $\{x^{-2}, x^{-1}, 1 + x^{-2}y, x + x^{-1}y\} \pmod f$

$\implies \langle (x^{-2}, 0), (x^{-1}, 0), (1, x^{-2}), (x, x^{-1}) \rangle_{\mathbb{F}} \subset \mathbb{F}((x^{-1}))^2$



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P/Q-digit systems in $\mathbb{F}[x]$

- Given:
  - $\mathbb{F} = \mathbb{F}_q$ a finite field of order q
  - $P, Q \in \mathbb{F}[x]$ coprime with $\deg P > \deg Q > 0$
  - $w \in \mathbb{F}[x]$
- *P/Q-polynomial digit expansion of w :*

$$w = \sum_{i=0}^{k} \frac{s_i}{Q} \left(\frac{P}{Q} \right)^i$$

where $k \geq 0$ and $s_i \in \mathcal{D} := \{ s \in \mathbb{F}[x] \mid \deg s < \deg P \}$.

Theorem (Loquias-Mkaouar-Scheicher-Thuswaldner, 2017)

Let $P, Q \in \mathbb{F}[x]$ be coprime with $\deg P > \deg Q > 0$. Then every element of $\mathbb{F}[x]$ admits a unique P/Q-polynomial digit expansion.

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Examples

Example

Given $\mathbb{F} = \mathbb{F}_2$, $P = x^2 + 1$, $Q = x$

$$x^2 + x = \frac{x}{Q}\left(\frac{P}{Q}\right)^2 + \frac{x+1}{Q}\left(\frac{P}{Q}\right) + \frac{x+1}{Q}$$

$$x^3 + 1 = \frac{x}{Q}\left(\frac{P}{Q}\right)^3 + \frac{1}{Q}\left(\frac{P}{Q}\right)^2 + \frac{0}{Q}\left(\frac{P}{Q}\right) + \frac{x+1}{Q}$$

P/Q-digit systems in $\mathbb{F}[x]$

- *expansion tree* $T(P/Q)$: edge-labeled infinite directed graph $(\mathbb{F}[x], E)$ where $(v, w) \in E$ with label $s \in \mathcal{D}$ if and only if

$$w = \frac{Pv + s}{Q}.$$

Theorem (Loquias-Mkaouar-Scheicher-Thuswaldner, 2017)

$T(P/Q)$ is $q^{\deg P - \deg Q}$-regular with a loop at the root 0.

- *ω-language $W_{P/Q}$ of $T(P/Q)$* : set of all right-infinite strings formed from the labels of infinite paths starting from the root of $T(P/Q)$

Theorem (Loquias-Mkaouar-Scheicher-Thuswaldner, 2017)

The only eventually periodic string in $W_{P/Q}$ is $0^\omega = 00\cdots$.

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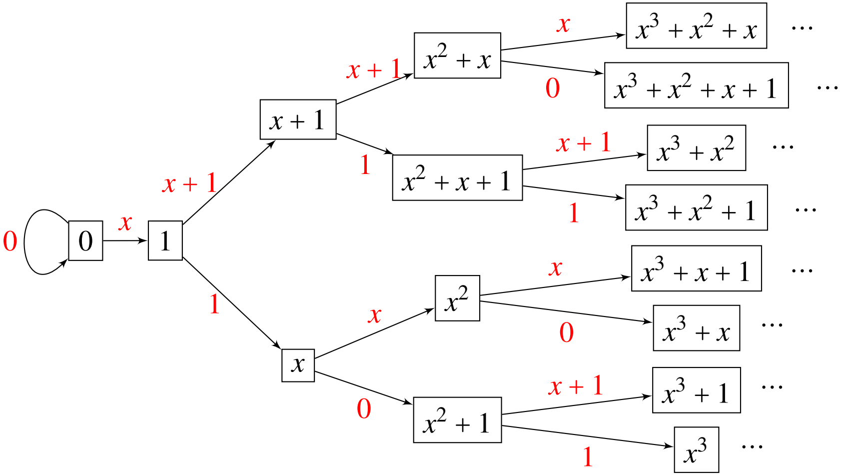
The expansion tree $T(P/Q)$ for $P = x^2 + 1$, $Q = x \in \mathbb{F}_2[x]$



P/Q-digit systems in $\mathbb{F}((x^{-1}))$

- Given:
  - $\mathbb{F} = \mathbb{F}_q$ a finite field of order q
  - $P, Q \in \mathbb{F}[x]$ coprime with $\deg P > \deg Q > 0$
  - $\alpha \in \mathbb{F}((x^{-1}))$
- *P/Q-series digit expansion of α :*

$$\alpha = \sum_{i=-\infty}^{k} \frac{s_i}{Q}\left(\frac{P}{Q}\right)^i$$

where $k \in \mathbb{Z}$ and $s_k s_{k-1} \cdots \in W_{P/Q}$

Theorem (Loquias-Mkaouar-Scheicher-Thuswaldner, 2017)

Let $P, Q \in \mathbb{F}[x]$ be coprime with $\deg P > \deg Q > 0$. Then every element of $\mathbb{F}((x^{-1}))$ admits a unique P/Q-series digit expansion.

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Example

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Given $\mathbb{F} = \mathbb{F}_2$, $P = x^2 = 1$, $Q = x$

$\langle\!\langle x^2+x \rangle\!\rangle = x, x+1, x+1 \bullet 0, 1, x+1, x+1, 0, x, x+1, x+1, 0, 0, 0, 0, 0, x, \ldots$

$\langle\!\langle x^3+1 \rangle\!\rangle = x, 1, 0, x+1 \bullet 1, x, 0, x+1, x, 1, 0, x+1, 0, 0, 0, 0, x, 1, 0, x+1, \ldots$

Outline

Connecting P/Q-digit systems to tilings over Laurent series

- $f = Qy - P$ (non-monic in y if $\deg Q > 0$)
- $s \in S := \mathbb{F}((x^{-1}, y^{-1}))/f\mathbb{F}((x^{-1}, y^{-1}))$
- x-digit representation of s : $\alpha \in \mathbb{F}((x^{-1}))$ such that $s = \alpha + f\mathbb{F}((x^{-1}, y^{-1}))$
- *y-digit representation of s :*

$$d_y(\alpha) := \sum_{i=-\infty}^{k} \frac{s_i}{Q} y^i$$

where

$$\alpha = \sum_{i=-\infty}^{k} \frac{s_i}{Q} \left(\frac{P}{Q}\right)^i$$

is the P/Q-series digit expansion of α.

Connecting P/Q-digit systems to tilings over Laurent series

Fix $m = \deg P$, $n = \deg Q$.

- Consider the vector space $\mathcal{V}_y = \operatorname{span}_{\mathbb{F}((y^{-1}))} \left\{ \frac{x^0}{Q}, \ldots, \frac{x^{m-1}}{Q} \right\}$
- Let $E := \{ d_y(\alpha) \mid \alpha \in \mathbb{F}((x^{-1})) \}$

Proposition (L.-R., 20xx)

E is a proper subspace of $\mathcal{V}_y$ with dimension $m - n$ over $\mathbb{F}((y^{-1}))$. Moreover, a basis for E is any set $\{d_y(\alpha_1), \ldots, d_y(\alpha_{m-n})\}$ where $\alpha_1, \ldots, \alpha_{m-n} \in \mathbb{F}((x^{-1}))$ such that the left-most digits of the digit strings $\langle\!\langle \alpha_i \rangle\!\rangle_{P/Q}$ are linearly independent over $\mathbb{F}$.

Examples

Example

- Given $P = x^2 + 1$, $Q = x \in \mathbb{F}_2[x]$
- $B = \left\{ d_y \left(1 + \frac{1}{x^2} \right) \right\}$
- $\langle\!\langle 1 + \frac{1}{x^2} \rangle\!\rangle_{P/Q} = x \bullet 1, 0, 1, 0, 0, 0, 1, s_{-8}, s_{-9}, \ldots$ where

$$s_{-i} := \begin{cases} 1 & \text{if } i = 2^n - 1 \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore E = \operatorname{span}_{\mathbb{F}((y^{-1}))} \left\{ \left(\sum_{n=1}^{\infty} y^{1-2^n} \right) \left(\frac{x^0}{Q} \right) + (1) \left(\frac{x^1}{Q} \right) \right\}$$

Examples

Example

- Given $P = x^3 + 1$, $Q = x \in \mathbb{F}_2[x]$
- $B = \{d_y(\alpha_1), d_y(\alpha_2)\}$
- $$\langle\langle \alpha_1 \rangle\rangle_{P/Q} = x \bullet 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, \ldots$$
  $$\langle\langle \alpha_1 \rangle\rangle_{P/Q} = x^2 \bullet 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, \ldots$$

$$\therefore d_y(\alpha_1) = (y^{-1} + y^{-4} + y^{-7} + y^{-13} + \cdots)\left(\frac{x^0}{Q}\right) + (1)\left(\frac{x^1}{Q}\right) + (0)\left(\frac{x^2}{Q}\right)$$

$$\therefore d_y(\alpha_2) = (y^{-2} + y^{-8} + y^{-14} + \cdots)\left(\frac{x^0}{Q}\right) + (0)\left(\frac{x^1}{Q}\right) + (1)\left(\frac{x^2}{Q}\right)$$

Fundamental Domains for P/Q-digit systems

- *y-height of s* : $\mathrm{hgt}_y(s) = \deg_y(d_y(\alpha))$
- *y-norm* : $|s|_y := q^{\mathrm{hgt}_y(s)}$ where $q = \begin{cases} \#\mathbb{F} & \text{if } \mathbb{F} \text{ is finite} \\ e & \text{if } \mathbb{F} \text{ is infinite} \end{cases}$
- S is a complete topological $\mathbb{F}((y^{-1}))$-vector space wrt $|\cdot|_y$
- $S \cong E$ as $\mathbb{F}((y^{-1}))$-vector spaces
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$\mathcal{F}_x = \mathcal{F}_y$

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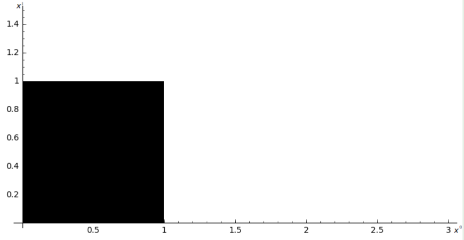

Recommendations for future work

- Find suitable digit systems for $\mathbb{F}((x^{-1},y^{-1}))/f\mathbb{F}((x^{-1},y^{-1}))$ where f is non-monic in at least one indeterminate and non-linear in both indeterminates
- Find suitable digit systems for $\mathbb{F}((x^{-1},y^{-1}))/f\mathbb{F}((x^{-1},y^{-1}))$ where f is non-monic in both indeterminates

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