Open maps: small and large holes

Nikita Sidorov

The University of Manchester

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It seems that a more immediate issue here is the "size" of an exclusion set. It looks plausible that if H is "large", then $\mathcal{J}(H)$ can contain only fixed points of f. On the other hand, if it is "small", then one would probably expect the Hausdorff dimension of $\mathcal{J}(H)$ to be positive.

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1. *We always have* $\dim_H \mathcal{J}(a,b) > 0$ *if*

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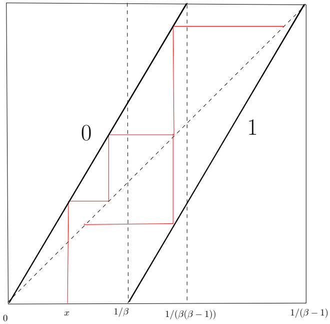

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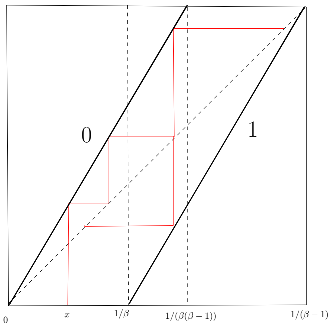

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Thus, $\mathcal{U}_\beta$ is the survivor set for T_β with the hole $[1/\beta, 1/(\beta(\beta - 1))]$.

The baker's map

Let $S : [0,1)^2 \to [0,1)^2$ denote the baker's map, i.e.

$$B(x,y) = \left(2x \bmod 1, \begin{cases} \frac{1}{2}y, & x < \frac{1}{2}, \\ \frac{1}{2}y + \frac{1}{2}, & x > \frac{1}{2} \end{cases} \right).$$

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As is well known, S is conjugate mod 0 to the two-sided shift on $\{0,1\}^{\mathbb{Z}}$ via the map π:

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Thus, in 2D the situation is completely different: there exist arbitrarily large convex holes whose survivor sets have positive Hausdorff dimension.

Definition

We will say that an open set H is a complete trap if $\mathcal{J}(H)$ does not contain any points except, possibly, of the form $\pi(\ldots 11110000 \ldots)$ or $\pi(\ldots 00001111 \ldots)$.

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2. Kevin Hare and I have generalised this result to all mixing subshifts (2017).

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If a convex hole H *has Lebesgue measure less than* 0.11802*, then* $\dim_H \mathcal{J}(H) > 0$*.*

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Thus, when we narrow our class of holes down to the convex holes, our initial intuition proves to be correct.

Finally, consider the following **disconnected** hole:
$\mathcal{H} = \left\{(x, y) : |y - x| > \frac{1}{2}\right\}$.

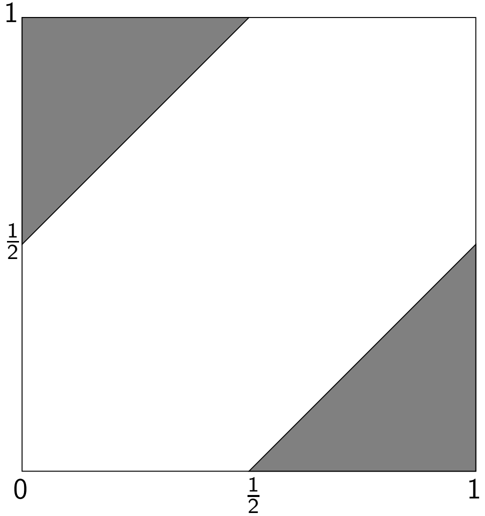

Figure: The hole $\mathcal{H}$

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