The level of distribution of the Thue–Morse sequence

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Section 1

The Thue–Morse sequence

The Thue–Morse sequence

(1) Start with $\mathbf{t}^{(0)} = 0$ and let $\mathbf{t}^{(k+1)}$ be the concatenation of $\mathbf{t}^{(k)}$ and its Boolean complement $\overline{\mathbf{t}^{(k)}}$.

$$\begin{aligned}
\mathbf{t}^{(0)} &= 0\\
\mathbf{t}^{(1)} &= 01\\
\mathbf{t}^{(2)} &= 0110\\
\mathbf{t}^{(3)} &= 01101001\\
\mathbf{t}^{(4)} &= 0110100110010110\\
\mathbf{t}^{(5)} &= 01101001100101101001011001101001
\end{aligned}$$

The Thue–Morse sequence $\mathbf{t}$ is the pointwise limit of this sequence.

(2) By induction, it follows that $\mathbf{t}$ is the fixed point of the substitution

$$0 \mapsto 01, \quad 1 \mapsto 10$$

that starts with 0.

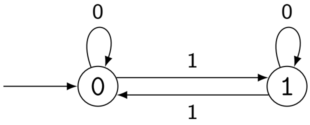
The Thue–Morse sequence, continued

(3) A third description uses the binary sum of digits function s:

$$s(\varepsilon_0 2^0 + \cdots + \varepsilon_\nu 2^\mu) = \varepsilon_0 + \cdots + \varepsilon_\nu \text{ for } \varepsilon_i \in \{0, 1\}.$$

We have $\mathbf{t}(n) = 0$ if and only if $s(n) \equiv 0 \bmod 2$. In the proofs, we work with $(-1)^{s(n)}$.

(4) The Thue–Morse sequence is one of the simplest automatic sequences:

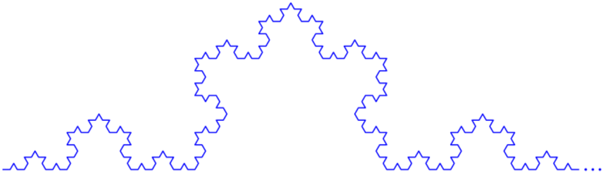


We feed in the binary expansion of n and obtain a letter $\in \{0, 1\}$.

The Thue–Morse sequence, continued

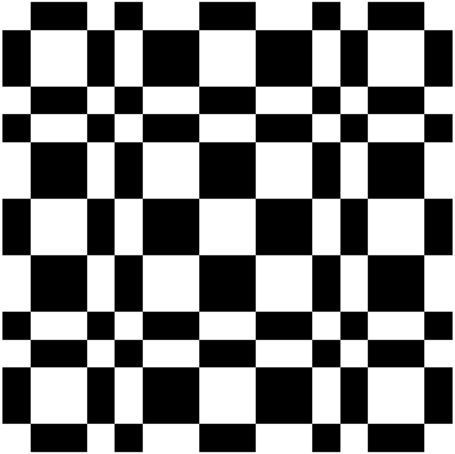
A less well-known characterization uses the Koch snowflake curve.

(5) The sequence $n \mapsto (-1)^{s(n)}\,\mathrm{e}(-n/3)$ describes the orientation of the nth segment in the unscaled snowflake curve (where $\mathrm{e}(x) = e^{2\pi i x}$):



The snowflake curve is the Thue–Morse sequence in disguise.

Thue–Morse, 16×16.



The Thue–Morse sequence has low subword complexity: if $p(L)$ denotes the number of (contiguous) subwords of length L, then $p(L) \leq CL$ for some constant C. This is true for any automatic sequence. Here $C \leq 8$.

Thue–Morse along arithmetic progressions

$$\overset{\downarrow}{0}11\overset{\downarrow}{0}10\overset{\downarrow}{0}11\overset{\downarrow}{0}01\overset{\downarrow}{0}11\overset{\downarrow}{0}10\overset{\downarrow}{0}10\overset{\downarrow}{1}10\overset{\downarrow}{0}11\overset{\downarrow}{0}10\overset{\downarrow}{0}11\overset{\downarrow}{0}01\overset{\downarrow}{0}11\overset{\downarrow}{0}01\overset{\downarrow}{1}01\overset{\downarrow}{0}01\overset{\downarrow}{0}11\overset{\downarrow}{0}10\ldots$$

Theorem (Gelfond 1968)

Let $d \geq 1$ and a be integers. There is an absolute $\lambda < 1$ such that

$$\left|\{1 \leq n \leq x : \mathbf{t}(n) = 0, n \equiv a \bmod d\}\right| = \frac{x}{2d} + \mathcal{O}\left(x^{\lambda}\right).$$

That is, for all $d \geq 1$ and $a < d$ we have

$$\sum_{1 \leq m \leq M} (-1)^{s(md+a)} \leq CM^{\lambda}.$$

The Thue–Morse sequence shows *cancellation* along arithmetic progressions.

The error term for sums over APs

- However, we cannot not get a uniform constant C for all d and a. This is the case because there are arbitrarily long arithmetic progressions on which $\mathbf{t}$ is constant.
- Two ways to see this:
  1. Let $d = 2^\lambda + 1$. Then $2 \mid s(md)$ for all $m < 2^\lambda$.
  2. van der Waerden's theorem.
- Therefore we look at a certain average over d.

Section 2

The level of distribution

The averaged error term

Theorem (Fouvry–Mauduit 1996)

selects the "worst" AP

$$\sum_{1\le d\le D}\ \max_{\substack{y,z\\ z-y\le x}}\ \max_{0\le a<d}\left|\sum_{\substack{y\le n<z\\ n\equiv a \bmod d}}(-1)^{s(n)}\right| \le Cx^{1-\eta}$$

for some $\eta > 0$ *and* $D = x^{0.5924}$.

observed − expected

- The number 0.5924 is a *level of distribution* of the Thue–Morse sequence.
- Note that we have "trivial" summands (of size $\asymp x/d$) for $d = 2^\lambda + 1$, where $x \le 2^{2\lambda}$. These don't matter in the sum.

The level of distribution of the Thue–Morse sequence

Theorem (S. 2018+)

The Thue–Morse sequence has level of distribution 1*. More precisely, let* $0 < \varepsilon < 1$*. There exist* $\eta > 0$ *and* C *such that*

$$\sum_{1\le d\le D} \max_{\substack{y,z\\ z-y\le x}} \max_{0\le a<d} \left| \sum_{\substack{y\le n<z\\ n\equiv a \bmod d}} (-1)^{s(n)} \right| \le C x^{1-\eta}$$

for $D = x^{1-\varepsilon}$*.*

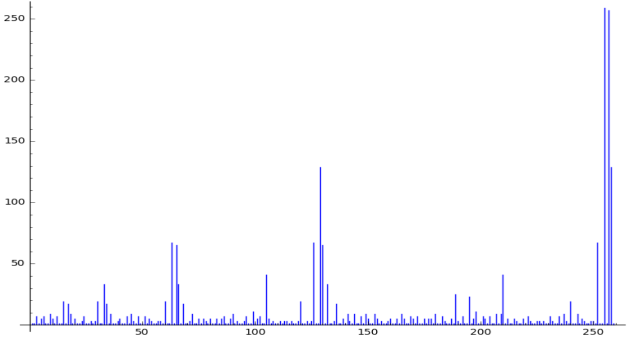
- This is a statement on *sparse* arithmetic progressions: the Thue–Morse sequence usually shows cancellation along N-term arithmetic progressions having common difference $\sim N^R$, where $R > 0$ is arbitrary ($R \le 1.46$ for Fouvry–Mauduit).

Open problem. What about $D = x(\log x)^{-A}$?

Corollary

For $d \geq 1, a \geq 0$ *define* $m_{d,a} = \min\{n : \mathbf{t}(nd + a) = 1\}$.
For each $\varepsilon > 0$ *we have*

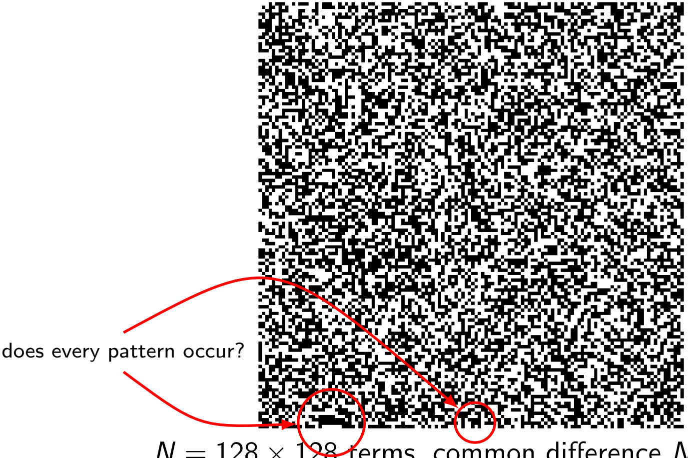
$$\left|\{d < D : \max_{a \geq 0} m_{d,a} \geq d^{\varepsilon}\}\right| = o(D).$$



That is, the first 1 in an arithmetic progression is not far away for most d.

Sparse arithmetic subsequences of $\mathbf{t}$

$\mathbf{t}$ along short arithmetic subsequences even seems to behave randomly.



$N = 128 \times 128$ terms, common difference $N^R = 3^{21}$

We know that arbitrarily long sequences of 0s appear as arithmetic subsequences of $\mathbf{t}$. What about other sequences?

Does every pattern occur?

Theorem (Müllner, S. 2017)

Every finite sequence over $\{0, 1\}$ *appears as an arithmetic subsequence of the Thue–Morse sequence.*

Added after the talk: Avgustinovich, Fon-Der-Flaass, and Frid proved this in 2000! $\rightsquigarrow$ search for "arithmetical complexity".

However: for given d and a, there are always blocks that do not occur in $\mathbf{t}(nd + a)$, since the subword complexity is at most linear!
$\rightsquigarrow$ consider sparse infinite subsequences of $\mathbf{t}$.

Section 3

Sparse infinite subsequences of Thue–Morse

Subsequences of $\mathbf{t}$

We wish to study $\mathbf{t}$ along subsequences of asymptotic density 0 and show (simple) normality of such subsequences (Knuth's R1 and R2).
Candidates:

- Polynomials with values in $\mathbb{N}$ } Gelfond problems
- Prime numbers } Gelfond problems
- 3^n
- $\lfloor f(n) \rfloor$ for f satisfying some growth conditions. For example, Piatetski-Shapiro sequences $\lfloor n^c \rfloor$.

Theorem (Corollary of Mauduit–Rivat 2009, Acta Math.)

The Thue–Morse sequence along the sequence of squares is simply normal.

Notorious open problem: $\mathbf{t}(n^3)$.

Theorem (Corollary of Mauduit–Rivat 2010, Ann. of Math.)

The Thue–Morse sequence along the primes is simply normal.

Subsequences of $\mathbf{t}$

- Related to $\mathbf{t}(3^n)$ is the following (hard) problem: is the sequence of fractional parts of $(3/2)^n$ dense in $[0,1]$?
  In both cases: study the binary expansion of $3^n = \sum_{0\le k\le n} \binom{n}{k} 2^k$.
- The sum of digits of $\lfloor n^c \rfloor$ is an approximation to the problem "the sum of digits of n^2", which could not be handled at first.

Theorem (Corollary of Mauduit–Rivat 2005)

Let $1 < c < 1.4$. *There exists an* $\eta > 0$ *such that*

$$\sum_{1\le n\le x} (-1)^{s(\lfloor n^c \rfloor)} \ll x^{1-\eta}.$$

In particular, for $1 < c < 1.4$, the sequence $n \mapsto \mathbf{t}(\lfloor n^c \rfloor)$ is simply normal.

The Thue–Morse sequence along sparse subsequences

Theorem (S. 2014)

The Thue–Morse sequence along $\lfloor n^c \rfloor$ *is simply normal for* $1 < c \le 1.42$.

Note that 1.42 is larger than 1.4 by "two cents".

Theorem (Drmota–Mauduit–Rivat 2018)

The Thue–Morse sequence along the sequence of squares is normal: every block $B \in \{0,1\}^k$ *appears as a subword with asymptotic frequency* $1/2^k$.

Theorem (S. 2015)

The Thue–Morse sequence along $\lfloor n^c \rfloor$ *is normal for* $1 < c < 4/3$.

Theorem (Müllner–S. 2017)

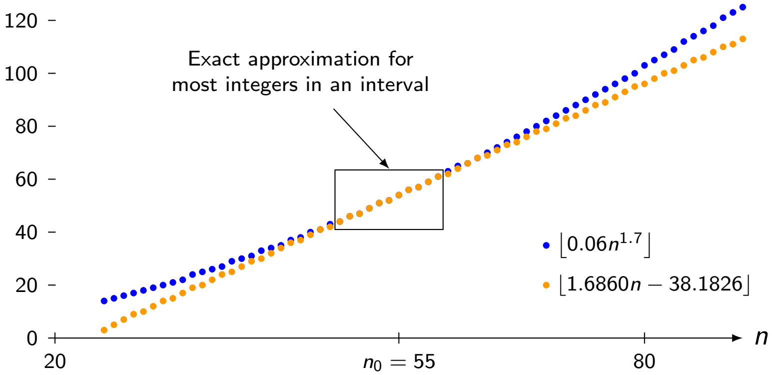
The Thue–Morse sequence along $\lfloor n^c \rfloor$ *is normal for* $1 < c < 1.5$.

Theorem (S. 2018+)

The Thue–Morse sequence along $\lfloor n^c \rfloor$ *is simply normal for* $1 < c < 2$.

Piatetski-Shapiro via Beatty sequences

The question on $\lfloor n^c \rfloor$ can be reduced to the linear case by Taylor approximation. This simplification comes at the cost of shorter sums.



For $c \to 2$, the slope $f'(n)$ is a large power of the length of the approximation interval.

Piatetski-Shapiro via Beatty sequences

Proposition (S. 2014)

We write $f(x) = x^c$, where $1 < c < 2$ is a real number. There exists a constant C such that for all $N \ge 2$ and $K > 0$ we have

$$\left| \frac{1}{N} \sum_{N < n \le 2N} (-1)^{s(\lfloor n^c \rfloor)} \right| \le C \left(f''(N) K^2 + \frac{(\log N)^2}{K} + \frac{J(f'(N), K)}{f'(N) K} \right),$$

$f'(N)$ a large power of K for c close to 2

where

$$J(D, K) = \int_D^{2D} \max_{\beta \ge 0} \left| \sum_{0 \le n < K} (-1)^{s(\lfloor n\alpha + \beta \rfloor)} \right| \mathrm{d}\alpha.$$

- The problem "simple normality" is reduced to a Beatty sequence version of our main theorem!
- The proof of this new statement is analogous to the main theorem.

Section 4

“Proof” of the main theorem

van der Corput's inequality

In the proof of the main theorem we make use of van der Corput's inequality:

Lemma

Let I be a finite interval containing N integers and let a_n be a complex number for $n \in I$. For all integers $K \geq 1$ and $R \geq 1$ we have

$$\left| \sum_{n \in I} a_n \right|^2 \leq \frac{N + K(R-1)}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R} \right) \sum_{\substack{n \in I \\ n+Kr \in I}} a_{n+Kr} \overline{a_n}.$$

Instead of the original sum, we now have to estimate certain correlations (where KR will be small compared to N).

Reducing the number of significant digits

- Iterating van der Corput's inequality, we introduce "multiple correlations", while reducing step by step the number of digits that have to be considered.
- We have to estimate

$$\sum_{r_1,\ldots,r_m<2^\rho} \sum_{0\le n<2^\rho} \mathrm{e}\left(\frac{1}{2} \sum_{\varepsilon_1,\ldots,\varepsilon_m\in\{0,1\}} s_\rho\left(n + \sum_{1\le i\le m} \varepsilon_i r_i\right)\right)$$

- This is a *Gowers norm* for the Thue–Morse sequence. An estimate of a very similar expression was given by Konieczny (2017). This finishes our "proof".

Thank you! [1]

[1] Supported by the FWF–ANR joint project MuDeRa