Some recent results related to Poissonian pair correlation problems

Wolfgang Stockinger

Johannes Kepler University

Institute for Financial Mathematics and Applied Number Theory

23. Mai 2018
Let \((x_n)_{n \in \mathbb{N}}\) be a sequence of real numbers in \([0, 1]\), and let \(\| \cdot \|\) denote the distance to the nearest integer. For every interval \([-s, s]\), we set

\[
R_2([-s, s], (x_n)_{n \in \mathbb{N}}, N) := \frac{1}{N} \# \left\{ 1 \leq l \neq m \leq N : \|x_l - x_m\| \leq \frac{s}{N} \right\}.
\]

A sequence \((x_n)_{n \in \mathbb{N}}\) in \([0, 1]\) is said to have Poissonian pair correlations, if for each \(s \geq 0\), \(R_2([-s, s], (x_n)_{n \in \mathbb{N}}, N)\) tends to \(2s\) as \(N \to \infty\).
Basic facts

- For random numbers $X_1, X_2, \ldots$ chosen uniformly and independently,

$$R_2([-s, s], (X_n)_{n \in \mathbb{N}}, N) \to 2s,$$

with probability tending to 1 as $N \to \infty$.

- Metrical theory of $(\{a_n \alpha\})_{n \in \mathbb{N}}$, where $(a_n)_{n \in \mathbb{N}}$ is an integer sequence, is well-understood.

- Kronecker sequences $(\{n \alpha\})_{n \in \mathbb{N}}$ are not Poissonian for any $\alpha$. 
Theorem (Grepstad-Larcher, Aistleitner-Lachmann-Pausinger, Steinerberger)

Let \((x_n)_{n \in \mathbb{N}}\) be a sequence in \([0, 1]\) and assume that for each \(s \geq 0\) the pair correlation function \(R_2([-s, s], (x_n)_{n \in \mathbb{N}}, N)\) tends to \(2s\) as \(N \to \infty\), then the sequence is uniformly distributed.
Let \((x_n)_{n \in \mathbb{N}}\) be a sequence in \([0, 1)\) with the following property: There is an \(s \in \mathbb{N}\), positive real numbers \(K\) and \(\gamma\), and infinitely many \(N\) such that the point set \(x_1, \ldots, x_N\) has a subset with \(M \geq \gamma N\) elements, denoted by \(x_{j_1}, \ldots, x_{j_M}\), which are contained in a set of points with cardinality at most \(KN\) having at most \(s\) different distances between neighbouring sequence elements, so-called gaps. Then, \((x_n)_{n \in \mathbb{N}}\) does not have Poissonian pair correlations.
Applications of Gap-Theorem

Sequences having a sufficiently large intersection with finite gap sequences (e.g., Kronecker sequences \(\{n\alpha\}\) for \(n \in \mathbb{N}\), ...)

**Corollary**

Let \((x_n)_{n \in \mathbb{N}}\) be a sequence in \([0,1)\) with the following property: There is a constant \(\kappa > 0\), a sequence \(N_1 < N_2, \ldots\) of positive integers and for each \(N_i, i \geq 1\), a Kronecker sequence \((y_{n}^{(i)})_{n \in \mathbb{N}}\) such that

\[
|\{x_1, \ldots, x_{N_i}\} \cap \{y_{1}^{(i)}, \ldots, y_{N_i}^{(i)}\}| \geq \kappa N_i,
\]

then \((x_n)_{n \in \mathbb{N}}\) does not have Poissonian pair correlations.

**Corollary**

If \((a_n)_{n \in \mathbb{N}}\) is quasi-arithmetic of degree \(d = 1\), then there is no \(\alpha\) such that the pair correlations of \(\{a_n \alpha\}\) for \(n \in \mathbb{N}\) are Poissonian.
Quasi-arithmetic of degree $d$

**Definition**

Let $(a_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of positive integers. We call this sequence *quasi-arithmetic of degree* $d$, where $d$ is a positive integer, if there exist constants $C, K > 0$ and a strictly increasing sequence $(N_i)_{i \geq 1}$ of positive integers such that for all $i \geq 1$ there is a subset $A^{(i)} \subset (a_n)_{1 \leq n \leq N_i}$ with $|A^{(i)}| \geq CN_i$ such that $A^{(i)}$ is contained in a $d$-dimensional arithmetic progression $P^{(i)}$ of size at most $KN_i$. 
Additive energy of a set of real numbers $A$ is defined to be

$$E(A) := \sum_{a+b=c+d} 1,$$

where the sum is extended over all quadruples $(a, b, c, d) \in A^4$. One has $|A|^2 \leq E(A) \leq |A|^3$.

**Theorem (Aistleitner-Larcher-Lewko)**

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of distinct integers and suppose that there exists a fixed constant $\epsilon > 0$ such that

$$E(A_N) \ll N^{3-\epsilon}, \quad N \to \infty,$$

where $A_N$ denotes the first $N$ elements of $(a_n)_{n \in \mathbb{N}}$. Then for almost all $\alpha$ one has

$$R_2([-s, s], \{a_n\alpha\}_{n \in \mathbb{N}}, N) \to 2s, \quad N \to \infty,$$

for all $s \geq 0$. 

Wolfgang Stockinger
23. Mai 2018 8 / 13
Is it possible for an increasing sequence of distinct integers \((a_n)_{n \in \mathbb{N}}\) which satisfies \(E(A_N) = \Omega(N^3)\) that the sequence \((\{a_n\alpha\})_{n \in \mathbb{N}}\) has Poissonian pair correlations for almost all \(\alpha\)?

If \((a_n)_{n \in \mathbb{N}}\) is an increasing sequence of distinct integers, does \(E(A_N) = o(N^3)\) imply that the sequence \((\{a_n\alpha\})_{n \in \mathbb{N}}\) has Poissonian pair correlations for almost all \(\alpha\)?
Theorem (Bourgain)

If \( E(A_N) = \Omega(N^3) \), where \( A_N \) denotes the first \( N \) elements of \( (a_n)_{n \in \mathbb{N}} \), then there exists a subset of \([0,1]\) of positive measure such that for every \( \alpha \) from this set the pair correlations of \( \{a_n\alpha\}_{n \in \mathbb{N}} \) are not Poissonian.

Theorem (Bourgain)

There exist sequences of distinct integers \( (a_n)_{n \in \mathbb{N}} \) with \( E(A_N) = o(N^3) \), such that \( \{a_n\alpha\}_{n \in \mathbb{N}} \) fails to have the metrical Poissonian pair correlation property.
Theorem (Lachmann-Technau)

Suppose that \((a_n)_{n \in \mathbb{N}}\) is a strictly increasing sequence of positive integers. If \(E(A_N) = \Omega(N^3)\), then the exceptional set has full Lebesgue measure.

Theorem (Aichinger-Aistleitner-Larcher)

For a strictly increasing sequence \((a_n)_{n \in \mathbb{N}}\) of positive integers we have \(E(A_N) = \Omega(N^3)\) if and only if \((a_n)_{n \in \mathbb{N}}\) is quasi-arithmetic of some degree \(d\).
Theorem (Larcher-S.)

If $E(A_N) = \Omega(N^3)$, then there is no $\alpha$ such that the pair correlations of $(a_n\alpha)_{n \in \mathbb{N}}$ are Poissonian.
Theorem (Hinrichs-Larcher-Ullrich)

Let \( x_1, x_2, \ldots \in [0, 1)^d \) be such that for every \( s \in \mathbb{N} \), and with

\[
    R^{(s)}_N := \frac{1}{N} \# \{ 1 \leq l \neq m \leq N | \| x_l - x_m \|_\infty \leq \frac{s}{N^{1/d}} \},
\]

we have

\[
    \lim_{N \to \infty} R^{(s)}_N = (2s)^d,
\]

then \( x_1, x_2, \ldots \) is uniformly distributed in \([0, 1)^d\).