Some recent results related to Poissonian pair correlation problems

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Poissonian Pair Correlations

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers in $[0,1]$, and let $\|\cdot\|$ denote the distance to the nearest integer. For every interval $[-s,s]$, we set

$$R_2([-s,s],(x_n)_{n\in\mathbb{N}},N) := \frac{1}{N}\#\left\{1 \le l \ne m \le N : \|x_l - x_m\| \le \frac{s}{N}\right\}.$$

A sequence $(x_n)_{n\in\mathbb{N}}$ in $[0,1]$ is said to have Poissonian pair correlations, if for each $s \geq 0$, $R_2([-s,s],(x_n)_{n\in\mathbb{N}},N)$ tends to $2s$ as $N \to \infty$.

Basic facts

- For random numbers $X_1, X_2, \ldots$ chosen uniformly and independently,

$$R_2([-s,s],(X_n)_{n\in\mathbb{N}},N) \to 2s,$$

  with probability tending to 1 as $N \to \infty$.

- Metrical theory of $(\{a_n\alpha\})_{n\in\mathbb{N}}$, where $(a_n)_{n\in\mathbb{N}}$ is an integer sequence, is well-understood.

- Kronecker sequences $(\{n\alpha\})_{n\in\mathbb{N}}$ are not Poissonian for any α.

PPC and uniform distribution modulo 1

Theorem (Grepstad-Larcher, Aistleitner-Lachmann-Pausinger, Steinerberger)

Let $(x_n)_{n\in\mathbb{N}}$ *be a sequence in* $[0,1]$ *and assume that for each* $s \geq 0$ *the pair correlation function* $R_2([-s,s],(x_n)_{n\in\mathbb{N}},N)$ *tends to* $2s$ *as* $N \to \infty$*, then the sequence is uniformly distributed.*

Gap-Theorem

Theorem (Larcher-S.)

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $[0,1)$ with the following property: There is an $s \in \mathbb{N}$, positive real numbers K and γ, and infinitely many N such that the point set $x_1, \ldots, x_N$ has a subset with $M \geq \gamma N$ elements, denoted by $x_{j_1}, \ldots, x_{j_M}$, which are contained in a set of points with cardinality at most KN having at most s different distances between neighbouring sequence elements, so-called gaps. Then, $(x_n)_{n \in \mathbb{N}}$ does not have Poissonian pair correlations.

Applications of Gap-Theorem

Sequences having a sufficiently large intersection with finite gap sequences (e.g., Kronecker sequences $(\{n\alpha\})_{n\in\mathbb{N}}, \dots$)

Corollary

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $[0,1)$ with the following property: There is a constant $\kappa > 0$, a sequence $N_1 < N_2, \dots$ of positive integers and for each N_i, $i \geq 1$, a Kronecker sequence $(y_n^{(i)})_{n\in\mathbb{N}}$ such that

$$|\{x_1, \dots, x_{N_i}\} \cap \{y_1^{(i)}, \dots, y_{N_i}^{(i)}\}| \geq \kappa N_i,$$

then $(x_n)_{n\in\mathbb{N}}$ does not have Poissonian pair correlations.

Corollary

If $(a_n)_{n\in\mathbb{N}}$ is quasi-arithmetic of degree $d = 1$, then there is no α such that the pair correlations of $(\{a_n\alpha\})_{n\in\mathbb{N}}$ are Poissonian.

Quasi-arithmetic of degree d

Definition

Let $(a_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of positive integers. We call this sequence *quasi-arithmetic of degree* d, where d is a positive integer, if there exist constants $C, K > 0$ and a strictly increasing sequence $(N_i)_{i\geq 1}$ of positive integers such that for all $i \geq 1$ there is a subset $A^{(i)} \subset (a_n)_{1\leq n\leq N_i}$ with $\left|A^{(i)}\right| \geq CN_i$ such that $A^{(i)}$ is contained in a d-dimensional arithmetic progression $P^{(i)}$ of size at most KN_i.

PPC and the additive energy

Additive energy of a set of real numbers A is defined to be

$$E(A) := \sum_{a+b=c+d} 1,$$

where the sum is extended over all quadruples $(a, b, c, d) \in A^4$. One has $|A|^2 \leq E(A) \leq |A|^3$.

Theorem (Aistleitner-Larcher-Lewko)

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of distinct integers and suppose that there exists a fixed constant $\epsilon > 0$ such that

$$E(A_N) \ll N^{3-\epsilon}, \qquad N \to \infty,$$

where A_N denotes the first N elements of $(a_n)_{n\in\mathbb{N}}$. Then for almost all α one has

$$R_2([-s, s], (\{a_n\alpha\})_{n\in\mathbb{N}}, N) \to 2s, \quad N \to \infty,$$

for all $s \geq 0$.

PPC and the additive energy

- Is it possible for an increasing sequence of distinct integers $(a_n)_{n\in\mathbb{N}}$ which satisfies $E(A_N) = \Omega(N^3)$ that the sequence $(\{a_n\alpha\})_{n\in\mathbb{N}}$ has Poissonian pair correlations for almost all α?

- If $(a_n)_{n\in\mathbb{N}}$ is an increasing sequence of distinct integers, does $E(A_N) = o(N^3)$ imply that the sequence $(\{a_n\alpha\})_{n\in\mathbb{N}}$ has Poissonian pair correlations for almost all α?

PPC and the additive energy

Theorem (Bourgain)

If $E(A_N) = \Omega(N^3)$, where A_N denotes the first N elements of $(a_n)_{n\in\mathbb{N}}$, then there exists a subset of $[0,1]$ of positive measure such that for ***every*** *α from this set the pair correlations of $(\{a_n\alpha\})_{n\in\mathbb{N}}$ are* ***not*** *Poissonian.*

Theorem (Bourgain)

There exist sequences of distinct integers $(a_n)_{n\in\mathbb{N}}$ with $E(A_N) = o(N^3)$, such that $(\{a_n\alpha\})_{n\in\mathbb{N}}$ fails to have the metrical Poissonian pair correlation property.

PPC and the additive energy

Theorem (Lachmann-Technau)

Suppose that $(a_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of positive integers. If $E(A_N) = \Omega(N^3)$, then the exceptional set has full Lebesgue measure.

Theorem (Aichinger-Aistleitner-Larcher)

For a strictly increasing sequence $(a_n)_{n\in\mathbb{N}}$ of positive integers we have $E(A_N) = \Omega(N^3)$ if and only if $(a_n)_{n\in\mathbb{N}}$ is quasi-arithmetic of some degree d.

Theorem (Larcher-S.)

If $E(A_N) = \Omega(N^3)$, then there is ***no*** *α such that the pair correlations of $(\{a_n\alpha\})_{n\in\mathbb{N}}$ are Poissonian.*

Higher-Dimensional concept

Theorem (Hinrichs-Larcher-Ullrich)

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \in [0,1)^d$ *be such that for every* $s \in \mathbb{N}$, *and with*

$$R_N^{(s)} := \frac{1}{N} \#\{1 \leq l \neq m \leq N | \ \|\boldsymbol{x}_l - \boldsymbol{x}_m\|_\infty \leq \frac{s}{N^{1/d}}\},$$

we have

$$\lim_{N \to \infty} R_N^{(s)} = (2s)^d,$$

then $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ *is uniformly distributed in* $[0,1)^d$.