Symmetric and congruent Rauzy fractals

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based on a joint-research with Klaus Scheicher and Víctor Sirvent
Motivation
Observation 1

**Congruence**

The Rauzy fractals induced by the substitutions

\[ \sigma : 1 \mapsto 1211, 2 \mapsto 311, 3 \mapsto 1 \quad \text{and} \quad \sigma' : 1 \mapsto 1112, 2 \mapsto 113, 3 \mapsto 1 \]

over the alphabet \( A = \{1, 2, 3\} \) are congruent (that is they differ by an affine transformation only).

The (original) Rauzy fractal induced by the substitutions

\[ \sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1 \]

(over the alphabet \( A = \{1, 2, 3\} \)) is central-symmetric with respect to some point \( c \).
Some Questions

- Which conditions ensure that Rauzy fractals are congruent?
Problem

Some Questions

- Which conditions ensure that Rauzy fractals are congruent?
- Which conditions ensure that a Rauzy fractal is central symmetric?
Symmetric Rauzy fractals

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Motivation
Construction
Congruence
Symmetry
Problems

Problem

Some Questions

- Which conditions ensure that Rauzy fractals are congruent?
- Which conditions ensure that a Rauzy fractal is central symmetric?
- What is the centre of symmetry?
Problem

Some Questions

- Which conditions ensure that Rauzy fractals are congruent?
- Which conditions ensure that a Rauzy fractal is central symmetric?
- What is the centre of symmetry?
- Are the conditions necessary?
Construction of the Rauzy fractal
Definitions

Notations

We denote
- $\mathcal{A}$ finite set (alphabet) (here $\mathcal{A} = \{1, 2, \ldots, m\}$)
- $\mathcal{A}^*$ set of finite words over $\mathcal{A}$
- $\varepsilon$ empty word
- $\tilde{X}$ mirror-word of $X \in \mathcal{A}^*$
- $|X|_y$ number of occurrences of the letter $y \in \mathcal{A}$ within the word $X \in \mathcal{A}^*$
- $l(X)$ “Abelianisation” of $X \in \mathcal{A}^*$, i.e.
  $l(X) = (|X|_1, \ldots, |X|_m) \in \mathbb{Z}^m$
Substitution and induced subspaces

Let $\sigma$ be a primitive unimodular Pisot substitution over $\mathcal{A}$, i.e. an endomorphism $\mathcal{A}^* \to \mathcal{A}^*$ such that $M_\sigma := (|\sigma(x)|_y)_{1 \leq x, y \leq m}$ is a primitive matrix; the dominant real eigenvalue $\theta > 1$ of $M_\sigma$ is a Pisot unit. Let $d + 1$ be the algebraic degree of $\theta$. If $d + 1 = m$ then $\sigma$ is irreducible. We define

- $E^u$ subspace spanned by the right eigenvector associated with $\theta$ ($E^u \cong \mathbb{R}$).
- $E^s$ subspace spanned by the right eigenvectors associated with the Galois conjugates different from $\theta$ ($E^u \cong \mathbb{R}^d$).
- $E^c$ subspace spanned by the right eigenvectors associated with the remaining eigenvalues ($E^u \cong \mathbb{R}^{m-d-1}$).
- $\pi$ projection of $\mathbb{R}^m$ onto $E^s$ (along $E^s$ and $E^c$).
Induced language and Rauzy fractal

**Definition**

Let \((x_j)_{j \geq 1} \in \mathcal{A}^\mathbb{N}\) be a periodic word (that is \(\sigma^n(x_1)\sigma^n(x_2)\sigma^n(x_3) \cdots = (x_j)_{j \geq 1}\) for some \(n \geq 1\)).

The language \(\mathcal{L}_\sigma\) induced by \(\sigma\) is the subset of words over \(\mathcal{A}\) that appear in \((x_j)_{j \geq 1}\), i.e.

\[
\mathcal{L}_\sigma = \{X \in \mathcal{A}^* : \exists 1 \leq i \leq j : X = x_i \cdots x_j\}.
\]
Induced language and Rauzy fractal

**Definition**

Let \((x_j)_{j \geq 1} \in A^\mathbb{N}\) be a periodic word (that is \(\sigma^n(x_1)\sigma^n(x_2)\sigma^n(x_3) \cdots = (x_j)_{j \geq 1}\) for some \(n \geq 1\)).

- The language \(\mathcal{L}_\sigma\) induced by \(\sigma\) is the subset of words over \(A\) that appear in \((x_j)_{j \geq 1}\), i.e.

\[
\mathcal{L}_\sigma = \{X \in A^*: \exists 1 \leq i \leq j: X = x_i \cdots x_j\}.
\]

- The Rauzy fractal associated with \(\sigma\) is the compact set

\[
\mathcal{R}_\sigma := \{\pi \circ \mathcal{I}(x_1 \cdots x_n) : n \in \mathbb{N}\} \subset E^s.
\]
On congruence
A general result

Theorem

Let $\sigma, \sigma'$ be irreducible primitive unimodular Pisot substitutions over the same alphabet $A$. If $\mathcal{L}_\sigma = \mathcal{L}_{\sigma'}$ then $\mathcal{R}_\sigma$ and $\mathcal{R}_{\sigma'}$ are congruent.
A general result

Theorem

Let $\sigma, \sigma'$ be irreducible primitive unimodular Pisot substitutions over the same alphabet $A$. If $\mathcal{L}_\sigma = \mathcal{L}_{\sigma'}$ then $R_\sigma$ and $R_{\sigma'}$ are congruent.

Remark

For reducible substitutions this does not hold in general. For example, the substitutions $\sigma_1, \sigma_2, \sigma_3$ over $A = \{1, 2, 3\}$ induce the same language, but ...
Conjugacy

Definition

Two substitutions $\sigma, \sigma'$ over $\mathcal{A}$ are *conjugated* (written $\sigma \sim \sigma'$) if there exists a word $X \in \mathcal{A}^*$ such that for each $y \in \mathcal{A}$ we have $X\sigma(y) = \sigma'(y)X$ (or for each $y \in \mathcal{A}$ we have $\sigma(y)X = X\sigma'(y)$).
Conjugacy

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Two substitutions $\sigma, \sigma'$ over $A$ are conjugated (written $\sigma \sim \sigma'$) if there exists a word $X \in A^*$ such that for each $y \in A$ we have $X\sigma(y) = \sigma'(y)X$ (or for each $y \in A$ we have $\sigma(y)X = X\sigma'(y)$).

Lemma
If two substitutions $\sigma, \sigma'$ over $A$ are conjugated then $L_\sigma = L_{\sigma'}$ and $M_\sigma = M_{\sigma'}$. 
Conjugacy

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Two substitutions $\sigma, \sigma'$ over $A$ are conjugated (written $\sigma \sim \sigma'$) if there exists a word $X \in A^*$ such that for each $y \in A$ we have $X\sigma(y) = \sigma'(y)X$ (or for each $y \in A$ we have $\sigma(y)X = X\sigma'(y)$).

Lemma

If two substitutions $\sigma, \sigma'$ over $A$ are conjugated then $\mathcal{L}_\sigma = \mathcal{L}_{\sigma'}$ and $M_\sigma = M_{\sigma'}$.

Theorem

Suppose that $\sigma \sim \sigma'$ such that $X\sigma(y) = \sigma'(y)X$ holds for all $y \in A$. Then $R_{\sigma'} = R_\sigma + t$ with $t = \sum_{n \geq 0} f^n \circ \pi \circ l(X) \in E^s$, where $f$ is the restriction of the action of $M_\sigma$ on $E^s$ (especially, $f$ is a contraction).
Example

Our initial example

The Rauzy fractals induced by the substitutions

\[ \sigma : 1 \mapsto 1211, 2 \mapsto 311, 3 \mapsto 1 \]  
and

\[ \sigma' : 1 \mapsto 1112, 2 \mapsto 113, 3 \mapsto 1 \]

over the alphabet \( A = \{1, 2, 3\} \) differ by a translation only since \( \sigma \sim \sigma' \) (we have \( 11\sigma(y) = \sigma'(y)11 \) for all \( y \in A \)). We can easily calculate the translation vector \( t \).
On symmetry
A general result

**Definition**
The language $\mathcal{L}_\sigma$ induced by a primitive substitution $\sigma$ is called *mirror-invariant* if for each $X \in \mathcal{L}_\sigma$ we have $\tilde{X} \in \mathcal{L}_\sigma$. 

**Theorem**
Let $\sigma$ be a primitive unimodular Pisot substitution such that the language $\mathcal{L}_\sigma$ is mirror-invariant. Then the Rauzy fractal $R_\sigma$ is central symmetric (with respect to some centre of symmetry $c$).

**Example**
The (reducible) substitution $\sigma: 1 \mapsto 23, 2 \mapsto 23, 3 \mapsto 45, 4 \mapsto 23, 5 \mapsto 1$ over $A = \{1, 2, 3, 4, 5\}$ induces the (original) Rauzy fractal which is central symmetric but $\mathcal{L}_\sigma$ is not mirror-invariant (the words of length 2 in $\mathcal{L}_\sigma$ are given by $\{12, 23, 31, 34, 45, 52\}$).
A general result

Definition

The language \( L_\sigma \) induced by a primitive substitution \( \sigma \) is called \textit{mirror-invariant} if for each \( X \in L_\sigma \) we have \( \tilde{X} \in L_\sigma \).

Theorem

Let \( \sigma \) be a primitive unimodular Pisot substitution such that the language \( L_\sigma \) is mirror-invariant. Then the Rauzy fractal \( R_\sigma \) is central symmetric (with respect to some centre of symmetry \( c \)).
Definition

The language $\mathcal{L}_\sigma$ induced by a primitive substitution $\sigma$ is called *mirror-invariant* if for each $X \in \mathcal{L}_\sigma$ we have $\tilde{X} \in \mathcal{L}_\sigma$.

Theorem

Let $\sigma$ be a primitive unimodular Pisot substitution such that the language $\mathcal{L}_\sigma$ is mirror-invariant. Then the Rauzy fractal $\mathcal{R}_\sigma$ is central symmetric (with respect to some centre of symmetry $c$).

Example

The (reducible) substitution

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Necessity

Definition

A primitive unimodular Pisot substitution $\sigma$ is said to have the tiling property if $R_\sigma$ induces a proper lattice tiling with respect to the lattice

$$\{\pi(z_1, \ldots, z_m) : (z_1, \ldots, z_m) \in \mathbb{Z}^m, z_1 + \cdots + z_m = 0\}.$$
Necessity

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**Conjecture (Pisot conjecture)**

Each irreducible primitive unimodular Pisot substitution has the tiling property.
Necessity

Definition
A primitive unimodular Pisot substitution $\sigma$ is said to have the tiling property if $R_\sigma$ induces a proper lattice tiling with respect to the lattice
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Conjecture (Pisot conjecture)
Each irreducible primitive unimodular Pisot substitution has the tiling property.

Theorem
Let $\sigma$ be a primitive unimodular Pisot substitution with central symmetric Rauzy fractal $R_\sigma$ that possesses the tiling property. Then the language $\mathcal{L}_\sigma$ is mirror-invariant.
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Substitutions that are conjugate to their mirror substitution

**Definition**

For a substitution $\sigma$ we define the *mirror-substitution* $\tilde{\sigma}$ by $\tilde{\sigma}(y) := \sigma(y)$ for each $y \in A$. 

**Theorem**

Let $\sigma$ be a primitive unimodular Pisot substitution such that $\sigma(y) = X = \tilde{\sigma}(y)$ holds for all $y \in A$. Then the Rauzy fractal $R_{\sigma}$ is central symmetric with respect to $c := \frac{1}{2} \sum_{n \geq 0} f_n \circ \pi \circ l(X)$. 
Substitutions that are conjugate to their mirror substitution

**Definition**
For a substitution $\sigma$ we define the *mirror-substitution* $\tilde{\sigma}$ by
$$\tilde{\sigma}(y) := \sigma(y)$$
for each $y \in \mathcal{A}$.

**Theorem**
Let $\sigma$ be a primitive unimodular Pisot substitution such that
$$\sigma(y)X = X\tilde{\sigma}(y)$$
holds for all $y \in \mathcal{A}$. Then the Rauzy fractal $\mathcal{R}_\sigma$ is central symmetric with respect to
$$c := \frac{1}{2} \sum_{n \geq 0} f^n \circ \pi \circ \mathbf{l}(X).$$
Arnoux-Rauzy substitutions

Definition

Let

\[ \sigma_1 : 1 \mapsto 1, 2 \mapsto 12, 3 \mapsto 13 \]
\[ \sigma_2 : 1 \mapsto 21, 2 \mapsto 2, 3 \mapsto 23 \]
\[ \sigma_3 : 1 \mapsto 31, 2 \mapsto 32, 3 \mapsto 3. \]

Each composition that includes \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) at least once is a primitive, irreducible, unimodular Pisot substitution.
Arnoux-Rauzy substitutions

Definition

Let

\[
\begin{align*}
\sigma_1 &: 1 \mapsto 1, 2 \mapsto 12, 3 \mapsto 13 \\
\sigma_2 &: 1 \mapsto 21, 2 \mapsto 2, 3 \mapsto 23 \\
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\end{align*}
\]

Each composition that includes \(\sigma_1\), \(\sigma_2\) and \(\sigma_3\) at least once is a primitive, irreducible, unimodular Pisot substitution.

Theorem

If \(\sigma = \sigma_{i_1} \circ \cdots \circ \sigma_{i_n}\) then \(\sigma(y)X = X\tilde{\sigma}(y)\) for all \(y \in \{1, 2, 3\}\) with

\[
X = \sigma_{i_1}(\sigma_{i_2}(\sigma_{i_3}(...(\sigma_{i_{n-1}}(i_n)i_{n-1})\cdots)i_3)i_2)i_1.
\]
A specific Arnoux-Rauzy substitution

Let $\sigma = \sigma_2 \circ \sigma_1 \circ \sigma_2 \circ \sigma_2 \circ \sigma_3$. Then for each $y \in \{1, 2, 3\}$ we have $\sigma(y)X = X\tilde{\sigma}(y)$ with

$$X = \sigma_2(\sigma_1(\sigma_2(3)2)2)1)2 = 2122122123212212212.$$ 

Therefore, $\mathcal{R}_\sigma$ is central symmetric with respect to $c := \frac{1}{2} \sum_{n \geq 0} f^n \circ \pi \circ I(X)$. 

![Graph of Rauzy fractal with point c]
Problems and open Questions
The class $\mathcal{P}$-conjecture

**Definition**

The language $\mathcal{L}_\sigma$ induced by a primitive substitution $\sigma$ is called *palindromic* if it contains infinitely many palindromes.
# The class $P$-conjecture

## Definition

The language $L_\sigma$ induced by a primitive substitution $\sigma$ is called **palindromic** if it contains infinitely many palindromes.


Let $\sigma$ be a primitive substitution such that $L_\sigma$ is palindromic. Then there exist a primitive substitution $\sigma'$ with $\sigma' \sim \tilde{\sigma}'$ (the class $P$) such that $L_\sigma = L_{\sigma'}$. 

Remark: The conjecture is solved for the 2-letter case (Tan: 2007) and for a class of substitutions related with interval exchange transformations (Masáková-Pelantová-Starosta: 2017).
The class $\mathcal{P}$-conjecture

Definition
The language $\mathcal{L}_\sigma$ induced by a primitive substitution $\sigma$ is called *palindromic* if it contains infinitely many palindromes.

Let $\sigma$ be a primitive substitution such that $\mathcal{L}_\sigma$ is palindromic. Then there exist a primitive substitution $\sigma'$ with $\sigma' \sim \tilde{\sigma}'$ (the class $\mathcal{P}$) such that $\mathcal{L}_\sigma = \mathcal{L}_{\sigma'}$.

Remark
The conjecture is solved for the 2-letter case (Tan: 2007) and for a class of substitutions related with interval exchange transformations (Masáková-Pelantová-Starosta: 2017).
The class $\mathcal{P}$-conjecture in context with symmetric Rauzy fractals

The example from above

The substitutions $\sigma_1, \sigma_2, \sigma_3$ over $A = \{1, 2, 3\}$ induce the same language which is palindromic, but only $\sigma_3$ is conjugate to its mirror-substitution.

<table>
<thead>
<tr>
<th>Substitution</th>
<th>Rauzy fractal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_1 : 1 \mapsto 131, 2 \mapsto 312, 3 \mapsto 2$</td>
<td>![Rauzy fractal for $\sigma_1$]</td>
</tr>
<tr>
<td>$\sigma_2 : 1 \mapsto 13, 2 \mapsto 1312, 3 \mapsto 12$</td>
<td>![Rauzy fractal for $\sigma_2$]</td>
</tr>
<tr>
<td>$\sigma_3 : 1 \mapsto 12, 2 \mapsto 1313, 3 \mapsto 13$</td>
<td>![Rauzy fractal for $\sigma_3$]</td>
</tr>
</tbody>
</table>
Proposition

A palindomic language is always mirror invariant.
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A palindomic language is always mirror invariant.

Question
Is there a primitive substitution $\sigma$ such that $\mathcal{L}_\sigma$ is mirror-invariant but not palindromic?
Palindomicity vs. mirror-invariance

**Proposition**
A palindomic language is always mirror invariant.

**Question**
Is there a primitive substitution $\sigma$ such that $L_\sigma$ is mirror-invariant but not palindromic?

**Partial answer**
In the two-letter case palindomicity and mirror-invariance are equivalent (Tan: 2007).
Thank you for your attention