Symmetric and congruent Rauzy fractals

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based on a joint-research with Klaus Scheicher and Víctor Sirvent

FWF Der Wissenschaftsfonds.

BOKU

Motivation

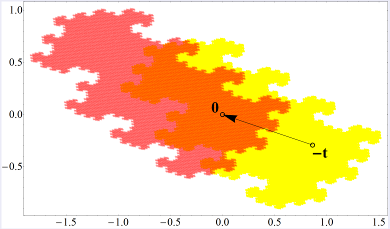
Observation 1

Congruence

The Rauzy fractals induced by the substitutions

$$\sigma : 1 \mapsto 1211, 2 \mapsto 311, 3 \mapsto 1 \text{ and}$$
$$\sigma' : 1 \mapsto 1112, 2 \mapsto 113, 3 \mapsto 1$$

over the alphabet $\mathcal{A} = \{1, 2, 3\}$ are congruent (that is they differ by an affine transformation only).



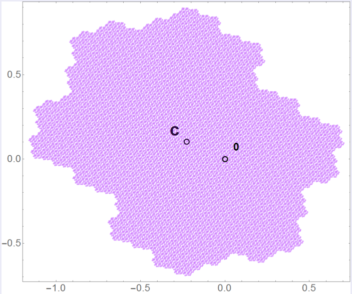
Observation 2

Symmetry (Sellami, Sirvent: 2011, 2012, 2016)

The (original) Rauzy fractal induced by the substitutions

$$\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$$

(over the alphabet $\mathcal{A} = \{1, 2, 3\}$) is central-symmetric with respect to some point **c**.



Problem

Some Questions

- Which conditions ensure that Rauzy fractals are congruent?

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- Which conditions ensure that Rauzy fractals are congruent?
- Which conditions ensure that a Rauzy fractal is central symmetric?
- What is the centre of symmetry?
- Are the conditions necessary?

Construction of the Rauzy fractal

Definitions

Notations

We denote

$\mathcal{A}$	finite set (alphabet) (here $\mathcal{A} = \{1, 2, \ldots, m\}$)
$\mathcal{A}^*$	set of finite words over $\mathcal{A}$
ε	empty word
$\tilde{X}$	mirror-word of $X \in \mathcal{A}^*$
$\lvert X \rvert_y$	number of occurrences of the letter $y \in \mathcal{A}$ within the word $X \in \mathcal{A}^*$
$\mathbf{l}(X)$	"Ablianisation" of $X \in \mathcal{A}^*$, *i.e.* $\mathbf{l}(X) = (\lvert X \rvert_1, \ldots, \lvert X \rvert_m) \in \mathbb{Z}^m$

Some linear algebra

Substitution and induced subspaces

Let σ be a primitive unimodular Pisot substitution over $\mathcal{A}$, *i.e.* an endomorphism $\mathcal{A}^* \longrightarrow \mathcal{A}^*$ such that $\mathbf{M}_\sigma := (|\sigma(x)|_y)_{1 \leq x,y \leq m}$ is a primitive matrix; the dominant real eigenvalue $\theta > 1$ of $\mathbf{M}_\sigma$ is a Pisot unit. Let $d+1$ be the algebraic degree of θ. If $d+1 = m$ then σ is irreducible. We define

- E^u — subspace spanned by the right eigenvector associated with θ ($E^u \cong \mathbb{R}$).
- E^s — subspace spanned by the right eigenvectors associated with the Galois conjugates different from θ ($E^u \cong \mathbb{R}^d$).
- E^c — subspace spanned by the right eigenvectors associated with the remaining eigenvalues ($E^u \cong \mathbb{R}^{m-d-1}$).
- π — projection of $\mathbb{R}^m$ onto E^s (along E^s and E^c)

Induced language and Rauzy fractal

Definition

Let $(x_j)_{j\geq 1} \in \mathcal{A}^{\mathbb{N}}$ be a periodic word (that is $\sigma^n(x_1)\sigma^n(x_2)\sigma^n(x_3)\cdots = (x_j)_{j\geq 1}$ for some $n \geq 1$).

- The language $\mathfrak{L}_\sigma$ induced by σ is the subset of words over $\mathcal{A}$ that appear in $(x_j)_{j\geq 1}$, *i.e.*

$$\mathfrak{L}_\sigma = \{X \in \mathcal{A}^* : \exists 1 \leq i \leq j : X = x_i \cdots x_j\}.$$

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- The Rauzy fractal associated with σ is the compact set

$$\mathcal{R}_\sigma := \overline{\{\pi \circ \mathbf{l}(x_1 \cdots x_n) : n \in \mathbb{N}\}} \subset E^s.$$

On congruence

A general result

Theorem

Let σ, σ' be irreducible primitive unimodular Pisot substitutions over the same alphabet $\mathcal{A}$. If $\mathfrak{L}_\sigma = \mathfrak{L}_{\sigma'}$ then $\mathcal{R}_\sigma$ and $\mathcal{R}_{\sigma'}$ are congruent.

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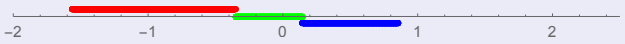
Remark

For reducible substitutions this does not hold in general. For example, the substitutions $\sigma_1, \sigma_2, \sigma_3$ over $\mathcal{A} = \{1, 2, 3\}$ induce the same language, but ...

Substitution	Rauzy fractal
$\sigma_1 : 1 \mapsto 131, 2 \mapsto 312, 3 \mapsto 2$	
$\sigma_2 : 1 \mapsto 13, 2 \mapsto 1312, 3 \mapsto 12$	
$\sigma_3 : 1 \mapsto 12, 2 \mapsto 1313, 3 \mapsto 13$	

Conjugacy

Definition

Two substitutions σ, σ' over $\mathcal{A}$ are *conjugated* (written $\sigma \sim \sigma'$) if there exists a word $X \in \mathcal{A}^*$ such that for each $y \in \mathcal{A}$ we have $X\sigma(y) = \sigma'(y)X$ (or for each $y \in \mathcal{A}$ we have $\sigma(y)X = X\sigma'(y)$).

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Lemma

If two substitutions σ, σ' over $\mathcal{A}$ are *conjugated* then $\mathfrak{L}_\sigma = \mathfrak{L}_{\sigma'}$ and $\mathbf{M}_\sigma = \mathbf{M}_{\sigma'}$.

Theorem

Suppose that $\sigma \sim \sigma'$ such that $X\sigma(y) = \sigma'(y)X$ holds for all $y \in \mathcal{A}$. Then $\mathcal{R}_{\sigma'} = \mathcal{R}_\sigma + \mathbf{t}$ with $\mathbf{t} = \sum_{n \geq 0} f^n \circ \pi \circ \mathbf{l}(X) \in E^s$, where f is the restriction of the action of $\mathbf{M}_\sigma$ on E^s (especially, f is a contraction).

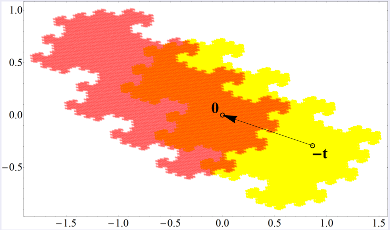
Example

Our initial example

The Rauzy fractals induced by the substitutions

$$\sigma : 1 \mapsto 1211, 2 \mapsto 311, 3 \mapsto 1 \text{ and}$$
$$\sigma' : 1 \mapsto 1112, 2 \mapsto 113, 3 \mapsto 1$$

over the alphabet $\mathcal{A} = \{1, 2, 3\}$ differ by a translation only since $\sigma \sim \sigma'$ (we have $11\sigma(y) = \sigma'(y)11$ for all $y \in \mathcal{A}$. We can easily calculate the translation vector $\mathbf{t}$.



On symmetry

A general result

Definition

The language $\mathfrak{L}_\sigma$ induced by a primitive substitution σ is called *mirror-invariant* if for each $X \in \mathfrak{L}_\sigma$ we have $\tilde{X} \in \mathfrak{L}_\sigma$.

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Theorem

Let σ be a primitive unimodular Pisot substitution such that the language $\mathfrak{L}_\sigma$ is mirror-invariant. Then the Rauzy fractal $\mathcal{R}_\sigma$ is central symmetric (with respect to some centre of symmetry $\mathbf{c}$).

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Example

The (reducible) substitution
$\sigma : 1 \mapsto 23, 2 \mapsto 23, 3 \mapsto 45, 4 \mapsto 23, 5 \mapsto 1$ over
$\mathcal{A} = \{1, 2, 3, 4, 5\}$ induces the (original) Rauzy fractal which is central symmetric but $\mathfrak{L}_\sigma$ is not mirror-invariant (the words of length 2 in $\mathfrak{L}_\sigma$ are given by $\{12, 23, 31, 34, 45, 52\}$).

Necessity

Definition

A primitive unimodular Pisot substitution σ is said to have the tiling property if $\mathcal{R}_\sigma$ induces a proper lattice tiling with respect to the lattice
$\{\pi(z_1, \ldots, z_m) : (z_1, \ldots, z_m) \in \mathbb{Z}^m, z_1 + \cdots + z_m = 0\}$.

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Conjecture (Pisot conjecture)

Each irreducible primitive unimodular Pisot substitution has the tiling property.

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Conjecture (Pisot conjecture)

Each irreducible primitive unimodular Pisot substitution has the tiling property.

Theorem

Let σ be a primitive unimodular Pisot substitution with central symmetric Rauzy fractal $\mathcal{R}_\sigma$ that possesses the tiling property. Then the language $\mathfrak{L}_\sigma$ is mirror-invariant.

Substitutions that are conjugate to their mirror substitution

Definition

For a substitution σ we define the *mirror-substitution* $\tilde{\sigma}$ by $\tilde{\sigma}(y) := \widetilde{\sigma(y)}$ for each $y \in \mathcal{A}$.

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Theorem

Let σ be a primitive unimodular Pisot substitution such that $\sigma(y)X = X\tilde{\sigma}(y)$ holds for all $y \in \mathcal{A}$. Then the Rauzy fractal $\mathcal{R}_\sigma$ is central symmetric with respect to $\mathbf{c} := \frac{1}{2}\sum_{n\geq 0} f^n \circ \pi \circ \mathbf{l}(X)$.

Arnoux-Rauzy substitutions

Definition

Let

$$\begin{aligned}\sigma_1 &: 1 \mapsto 1, 2 \mapsto 12, 3 \mapsto 13\\ \sigma_2 &: 1 \mapsto 21, 2 \mapsto 2, 3 \mapsto 23\\ \sigma_3 &: 1 \mapsto 31, 2 \mapsto 32, 3 \mapsto 3.\end{aligned}$$

Each composition that includes σ_1, σ_2 and σ_3 at least once is a primitive, irreducible, unimodular Pisot substitution.

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Theorem

If $\sigma = \sigma_{i_1} \circ \cdots \circ \sigma_{i_n}$ then $\sigma(y)X = X\tilde{\sigma}(y)$ for all $y \in \{1, 2, 3\}$ with

$$X = \sigma_{i_1}(\sigma_{i_2}(\sigma_{i_3}(\cdots(\sigma_{i_{n-1}}(i_n)i_{n-1})\cdots)i_3)i_2)i_1.$$

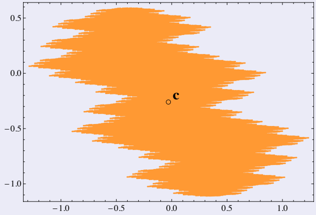
Example

A specific Arnoux-Rauzy substitution

Let $\sigma = \sigma_2 \circ \sigma_1 \circ \sigma_2 \circ \sigma_2 \circ \sigma_3$. Then for each $y \in \{1,2,3\}$ we have $\sigma(y)X = X\tilde{\sigma}(y)$ with

$$X = \sigma_2(\sigma_1(\sigma_2(\sigma_2(3)2)2)1)2 = 2122122123212212212.$$

Therefore, $\mathcal{R}_\sigma$ is central symmetric with respect to $\mathbf{c} := \frac{1}{2}\sum_{n\geq 0} f^n \circ \pi \circ \mathbf{l}(X)$.



Problems and open Questions

The class $\mathcal{P}$-conjecture

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Conjecture (Hof-Knill-Simon: 1995, Labbé: 2014, Harju-Vesti-Zamboni: 2015)

Let σ be a primitive substitution such that $\mathfrak{L}_\sigma$ is palindromic. Then there exist a primitive substitution σ' with $\sigma' \sim \tilde{\sigma}'$ (the class $\mathcal{P}$) such that $\mathfrak{L}_\sigma = \mathfrak{L}_{\sigma'}$.

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Remark

The conjecture is solved for the 2-letter case (Tan: 2007) and for a class of substitutions related with interval exchange transformations (Masáková-Pelantová-Starosta: 2017).

The class $\mathcal{P}$-conjecture in context with symmetric Rauzy fractals

The example from above

The substitutions $\sigma_1, \sigma_2, \sigma_3$ over $\mathcal{A} = \{1, 2, 3\}$ induce the same language which is palindromic, but only σ_3 is conjugate to its mirror-substitution..

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Palindomicity vs. mirror-invariance

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Partial answer

In the two-letter case palindomicity and mirror-invariance are equivalent (Tan: 2007).

Thanks

Thank you for your attention