Some news on rational self-affine tiles

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Integral self-affine tiles

Definition

Let A be an expanding $d \times d$ integer matrix and $\mathcal{D} \subset \mathbb{Z}^d$. The non-empty compact set $\mathcal{F} = \mathcal{F}(A, \mathcal{D})$ defined by

$$A\mathcal{F} = \bigcup_{d \in \mathcal{D}} (\mathcal{F} + d)$$

is called integral self-affine tile, if its d-dimensional Lebesgue-measure is positive.

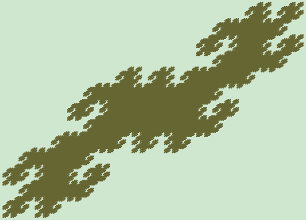
Integral self-affine tiles were studied extensively *e.g.* by R. Kenyon, K. Gröchenig, A. Haas, J. Lagarias, Y. Wang, and others

W.l.o.g. we always assume that $\mathbf{0} \in \mathcal{D}$.

Knuth's twin dragon

Example

$$A = \begin{pmatrix} 0 & -2 \\ 1 & -2 \end{pmatrix},$$
$$\mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$



This self-affine tile was studied by D. Knuth in connection with number systems defined in the ring $\mathbb{Z}[i]$.

Tilings

Lagarias and Wang proved that $\mathcal{F} + \mathbb{Z}^d$ forms a multiple tiling of $\mathbb{R}^d$. In particular $\lambda_d(\mathcal{F}) \in \mathbb{N}$.

Theorem (Lagarias and Wang 1997)

Let A be an integer matrix with irreducible characteristic polynomial and let $\mathcal{D}$ be a primitive standard digit set. Then $\mathcal{F}(A, \mathcal{D})$ tiles $\mathbb{R}^d$ with respect to the lattice $\mathbb{Z}^d$.

Proof: Based on a Fourier analytic tiling criterion of Gröchenig and Haas and on a result of Cerveau, Conze and Raugi on transfer operators.

In the case of reducible matrices the theorem is no longer true in general. Lagarias and Wang gave a characterization in this case.

Basic definitions

- For $A \in \mathbb{Q}^{d\times d}$ let

$$\mathbb{Z}^d[A] = \bigcup_{k\geq 0}(\mathbb{Z}^d + A\mathbb{Z}^d + \cdots + A^{k-1}\mathbb{Z}^d)$$

- We define a standard digit set $\mathcal{D}$ as a complete set of coset representatives of

$$\mathbb{Z}^d[A]/A\mathbb{Z}^d[A]$$

(without loss of generality, it can be assumed that $\mathbf{0} \in \mathcal{D}$).

- The pair $(A, \mathcal{D})$ is the data we need to define our main objects.

A^{-1}-adic stuff

- Let $\mathcal{E} \subset \mathbb{Z}^d$ be the set of coset representatives of
$$\mathbb{Z}^d[A^{-1}]/A^{-1}\mathbb{Z}^d[A^{-1}]$$
and set
$$\mathcal{E}_k := \mathbb{Z}^d[A^{-1}]/A^{-k}\mathbb{Z}^d[A^{-1}]$$
- A^{-1}-adic integers $\mathbb{Z}_{A^{-1}}$ as projective limit: $\mathbb{Z}_{A^{-1}} = \varprojlim \mathcal{E}_k$.
- A^{-1}-adic rationals are then is defined by $\mathbb{Q}_{A^{-1}} = \bigcup_{j\geq 0} A^j \mathbb{Z}_{A^{-1}}$;
- A^{-1}-adic norm: for $\mathbf{x} = \sum_{j=-\infty}^{k} A^j \mathbf{c}_j$ with $\mathbf{c}_k \neq \mathbf{0}$,
$$||\mathbf{x}||_{A^{-1}} = |\mathcal{E}|^k ,$$
- Haar measure $\mu_{A^{-1}}$ in $\mathbb{Q}_{A^{-1}}$, such that
$$\mu_{A^{-1}}(\mathbf{x} + A^k \mathbb{Z}_{A^{-1}}) = |\mathcal{E}|^k$$
for every $\mathbf{x} \in \mathbb{Q}_{A^{-1}}$.

Representation space: where the tiles live

- Representation space $\mathbb{K}_{A^{-1}} = \mathbb{R}^d \times \mathbb{Q}_{A^{-1}}$.
- The diagonal embedding

$$\Phi_{A^{-1}} : \mathbb{Z}^d[A, A^{-1}] \to \mathbb{K}_{A^{-1}}$$

  maps $\mathbf{x} \in \mathbb{Z}^d[A, A^{-1}]$ into $\Phi_{A^{-1}}(\mathbf{x}) = (\mathbf{x}, \mathbf{x})$.
- The multiplication by A in $\mathbb{K}_{A^{-1}}$: $A \cdot (\mathbf{x}, \mathbf{y}) = (A\mathbf{x}, A\mathbf{y})$.
- Product norm:

$$||\mathbf{x}|| := ||\mathbf{x}||_{\mathbb{K}} = \max\{||\mathbf{x}||_{\infty}, ||\mathbf{x}||_{A^{-1}}\}$$

- Product measure: $\mu_{\mathbb{K}} = \mu_{\infty} \otimes \mu_{A^{-1}}$
- By the definition of the measure $\mu_{\mathbb{K}}$, for every measurable set $\mathcal{M} \subset \mathbb{K}_{A^{-1}}$, one has

$$\mu_{\mathbb{K}}(A \cdot \mathcal{M}) = |\det A| \cdot |\mathcal{E}| \cdot \mu_{\mathbb{K}}(\mathcal{M}).$$

Rational self-affine tiles

The main object of our investigations is a compact set $\mathcal{F} = \mathcal{F}(A, \mathcal{D}) \subset \mathbb{K}_{A^{-1}}$ that satisfies the equation

$$A\mathcal{F} = \bigcup_{\mathbf{d} \in \mathcal{D}} (\mathcal{F} + (\mathbf{d}, \mathbf{d})).$$

The set $\mathcal{F}$ is called rational self-affine tile if $\mu_{\mathbb{K}}(\mathcal{F}) > 0$.

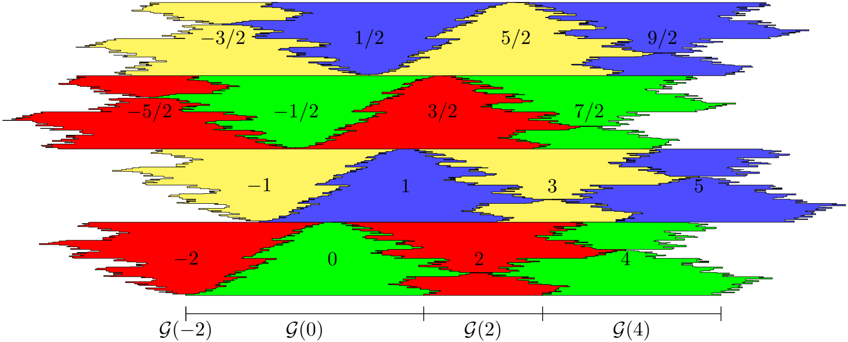
Lemma

The set $\Phi_{A^{-1}}(\mathbb{Z}^d[A])$ is a uniformly discrete and relatively dense subgroup of $\mathbb{K}_{A^{-1}}$.

An example ...

- $A = \frac{3}{2}$ and $\mathcal{D} = \{0, 1, 2\}$.
- Representation space $\mathbb{K}_{\frac{2}{3}} = \mathbb{R} \times \mathbb{Q}_2$.
- $\mathcal{F} = \mathcal{F}(\frac{3}{2}, \{0, 1, 2\})$ is a compact subset of $\mathbb{K}_{\frac{2}{3}}$,
- $\mathcal{F} = \overline{\mathrm{int}(\mathcal{F})}$ and $\mu_{\frac{3}{2}}(\partial\mathcal{F}) = 0$.
- $\{\mathcal{F} + \Phi_{\frac{2}{3}}(x) : x \in \mathbb{Z}[\frac{3}{2}]\}$, forms a tiling of $\mathbb{K}_{\frac{2}{3}}$.

... with a picture



Here, an element $\sum_{j=k}^{\infty} b_j \alpha^{-j}$ of $\mathbb{Q}_2$, with $b_j \in \{0,1\}$, is represented by $\sum_{j=k}^{\infty} b_j 2^{-j}$.

Properties of rational self-affine tiles

Theorem (Jankauskas, Steiner, and T. (2018+))

Let $A \in \mathbb{Q}^{d\times d}$ be an expanding rational matrix and let $\mathcal{D} \subset \mathbb{Z}^d[A]$ be a standard digit set. Then the following properties hold for the rational self-affine tile $\mathcal{F} = \mathcal{F}(A, \mathcal{D})$.

1) *$\mathcal{F}$ is a tile and $\mathcal{F}^{\mathrm{o}} \neq \varnothing$.*
2) *$\mathcal{F}$ is the closure of its interior, i.e. $\mathcal{F} = \overline{\mathcal{F}^{\mathrm{o}}}$.*
3) *The boundary $\partial\mathcal{F}$ of $\mathcal{F}$ has measure $\mu_{\mathbb{K}}(\partial\mathcal{F}) = 0$.*
4) *The collection of sets $\{\mathcal{F} + \Phi_{A^{-1}}(\mathbf{z}) : \mathbf{z} \in \mathbb{Z}^d[A]\}$ is a multi-tiling of $\mathbb{K}_{A^{-1}}$.*

"Adelic" definition of rational self affine tiles

Definition (Steiner and T. (2015))

Let α be a root of the characteristic polynomial and let (r, s) be the signature of $\mathbb{Q}(\alpha)$. Choose $\mathcal{D} \subset \mathbb{Z}[\alpha]$.

- Let $(\alpha) = \frac{\mathfrak{a}}{\mathfrak{b}}$ where $(\mathfrak{a}, \mathfrak{b}) = \mathcal{O}$ and define the representation space
$$\mathbb{K}_\alpha = \prod_{\mathfrak{p}|\infty \text{ or } \mathfrak{p}|\mathfrak{b}} K_\mathfrak{p} = \mathbb{R}^r \times \mathbb{C}^s \times \prod_{\mathfrak{p}|\mathfrak{b}} K_\mathfrak{p}.$$
- The rational self-affine tile $\mathcal{F} = \mathcal{F}_\alpha = \mathcal{F}(A, \mathcal{D})$ is defined by (if its Haar measure in $\mathbb{K}_\alpha$ is positive)
$$\alpha \mathcal{F}_\alpha = \bigcup_{d \in \mathcal{D}} (\mathcal{F}_\alpha + \Phi_\alpha(d)).$$

  Φ_α is the diagonal embedding of $\mathbb{Q}(\alpha)$ in $\mathbb{K}_\alpha$.

Remarks on the definition

Haar Measure

The Haar measure μ on $\mathbb{K}_\alpha$ is the product of the Haar measures $\mu_\mathfrak{p}$ on $K_\mathfrak{p}$. For finite $\mathfrak{p}$ we set $\mu_\mathfrak{p}(a+\mathfrak{p}^m) = \mathfrak{N}(\mathfrak{p})^{-m}$, for infinite primes $\mu_\mathfrak{p}$ is the Lebesgue measure on $K_\mathfrak{p}$ ($K_\mathfrak{p} = \mathbb{R}$ or $\mathbb{C}$).

Embedding of $\mathbb{Q}(\alpha)$

$\mathbb{Q}(\alpha)$ is naturally embedded in $\mathbb{K}_\alpha$ diagonally via $\Phi(\xi) = (\alpha, \ldots, \alpha)$. Moreover, $\mathbb{Q}(\alpha)$ acts multiplicatively on the ring $\mathbb{K}_\alpha$, in particular $\xi \cdot z = \Phi_\alpha(\xi) z$.

Tiling of $\mathbb{R}^d$

One could also define $\mathcal{F}_\alpha$ as a subset of $\mathbb{R}^d$ as in the integral case. However, this would result in sets that have no nice tiling properties.

Relation between the two definitions

- Choose an expanding matrix $A \in \mathbb{Q}^{d \times d}$ with irreducible characteristic polynomial.
- Let α be a root of the characteristic polynomial of A.
- Choose a basis of $\mathbb{Q}(\alpha)$ (viewed as a vector space over $\mathbb{Q}$) such that the multiplication by α is done by A in this vector space.
- Chose a subset $\mathcal{D} \subset \mathbb{Z}[\alpha]$ as set of digits.
- An equivalent definition for a rational self-affine tile $\mathcal{F} \subset \mathbb{K}_\alpha$ is given by the set equation

$$A\mathcal{F} = \bigcup_{d \in \mathcal{D}} (\mathcal{F} + \Phi_\alpha(d)).$$

Special digit sets

Again we need special properties of the digit set.

Definition

Let α be expanding and $\mathcal{D} \subset \mathbb{Z}[\alpha]$ be given.

- The digit set $\mathcal{D}$ is called primitive if

$$\langle \mathcal{D}, \alpha\mathcal{D}, \alpha^2\mathcal{D}, \ldots \rangle_{\mathbb{Z}} = \mathbb{Z}[\alpha].$$

- The digit set $\mathcal{D}$ is called standard digit set if $\mathcal{D}$ is a complete set of residues of $\mathbb{Z}[\alpha]/\alpha\mathbb{Z}[\alpha]$.

Set $\mathbb{Z}\langle \alpha, \mathcal{D} \rangle = \langle \mathcal{D}, \alpha\mathcal{D}, \alpha^2\mathcal{D}, \ldots \rangle_{\mathbb{Z}}$

Tiling theorems

Much harder to prove is the following tiling theorem.

Theorem (Steiner and T. (2015))

Let α be an expanding algebraic number and let $\mathcal{D}$ be a standard digit set for α. Then $\{\mathcal{F} + \Phi_\alpha(x) : x \in \mathbb{Z}\langle\alpha, \mathcal{D}\rangle\}$ forms a tiling of $\mathbb{K}_\alpha$.

For primitive digit sets, we get the following immediate corollary.

Corollary (Steiner and T. (2015))

Let α be an expanding algebraic number and let $\mathcal{D}$ be a primitive, standard digit set for α. Then $\{\mathcal{F} + \Phi_\alpha(x) : x \in \mathbb{Z}[\alpha]\}$ forms a tiling of $\mathbb{K}_\alpha$.

Note that for instance $\{0, 1\} \in \mathcal{D}$ implies primitivity of the digit set.

Thank you for your attention

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