Periodic representations in Salem bases

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$$\sum_{i=-k}^{+\infty} a_i \beta^{-i}, \quad a_i \in \mathcal{A}$$

A representation is called eventually periodic if $(a_i)_{i\in\mathbb{N}}$ is eventually periodic

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Theorem (V.)

Let β, $|\beta| > 1$, be an algebraic number, then there exists $\mathcal{A} \subset \mathbb{Z}$ such that every element of $\mathbb{Q}(\beta)$ admits an eventually periodic $(\beta, \mathcal{A})$-representation.

Theorem (K. Schmidt, 1980)

Let $\mathrm{Per}(\beta)$ *denote the set of numbers with eventually periodic (greedy)* β*-expansions. Then*

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Schmidt conjectured: $\mathrm{Per}(\beta) = \mathbb{Q}(\beta)$ holds for Salem numbers although a little is known (and that little is thanks to D. Boyd)

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Let β *be an algebraic number without conjugates on the unit circle. Then* $\mathbb{Q}(\beta)$ *admits eventually periodic representations with some (integer) alphabet.*

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If each $x \in \mathbb{Q}(\beta) \cap B_r(0)$ for some $r > 0$ has an eventually periodic representation of the form $x = \sum_{i \geq 1} a_i \beta^{-i}$, then $|\beta'| = |\beta|$ or $|\beta'| \leq 1$ holds for all the conjugates β' of β.

P. Savický, J. Šíma: Quasi-Periodic β-expansions and Cut Languages, Theoret. comp. sci., 2018

P. Savický: Eventually periodic representations for $\beta \approx 1.722084$ Salem root of $x^4 - x^3 - x^2 - x + 1$ and $\mathcal{A} = \{-2, -1, 0, 1, 2\}$

Representations in our case are made by the following procedure (Thurston 1989):

- let $V \subset \mathbb{C}$ be bounded
- let $\mathcal{A} \subset \mathbb{C}$ be finite
- let $T(x) = \beta x - D(x) : V \to V, \quad D(x) \in \mathcal{A}$

  (meaning that $\beta V \subset \bigcup_{a \in \mathcal{A}} (V + a)$)

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Greedy expansions: $V = [0, 1)$, $D(x) = \lfloor \beta x \rfloor$

If $x \notin V$, then find $n \in \mathbb{N}$ such that $\beta^{-n}x \in V$

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If $T^n(x) = T^m(x)$ for some $m, n \in \mathbb{N}$, then we get an eventually periodic representation

Pisot case

Let β be Pisot, β' its conjugate, i.e. $|\beta'| < 1$.

Take any $x \in \mathbb{Q}(\beta) \cap V$ and start iterating $T(x)$

We have:

- $T(x) = \beta x - a \in V$ in the identical embedding
- $|(T(x))'| = |\beta' x' - a'| < C$ eventually in other embeddings

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$(T^n(x))_{n\in\mathbb{N}}$ takes values in a lattice $\implies T^n(x) = T^m(x)$ eventually

Non-Pisot case

Two problems:

- If $|\beta'| \geq 1$, then $(T^n(x))'$ might be unbounded
- If β is a noninteger, then $T^n(x)$ does not take values in a lattice

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Solution: We define V bounded in embeddings where β is big, i.e.

- Embeddings corresponding to $|\beta'| \geq 1$
- $\mathfrak{p}$-adic embeddings where $|\beta|_{\mathfrak{p}} > 1$

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But if

$$\beta' B_1(0) \subset \bigcup_{a \in \mathcal{A}} (B_1(0) + a),$$

then

$$\beta' B_m(0) \subset \bigcup_{a \in \mathcal{A}} (B_m(0) + a)$$

holds for any $m > 1$ (with the same alphabet!)

General case

$$\beta^{-n}x \in V_m := \prod_{\substack{\mathfrak{p}|\infty \\ |\beta|_\mathfrak{p}>1}} B_1(0) \times \prod_{\substack{\mathfrak{p}|\infty \\ |\beta|_\mathfrak{p}=1}} B_m(0) \times \prod_{\substack{\mathfrak{p}\nmid\infty \\ |\beta|_\mathfrak{p}>1}} \mathcal{O}_\mathfrak{p} \subset \mathbb{K}_\beta$$

For each individual set $V_\mathfrak{p}$ in the product we can find $\mathcal{A}_\mathfrak{p}$ such that $\beta V_\mathfrak{p} \subset \bigcup_{a\in\mathcal{A}_\mathfrak{p}}(V_\mathfrak{p} + a)$

Then with $\mathcal{A} = \prod_\mathfrak{p} \mathcal{A}_\mathfrak{p}$ we have $\beta V_m \subseteq \bigcup_{a\in\mathcal{A}}(V_m + a)$

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Moreover, for (complex) Salem bases, all $x \in \mathbb{Q}(\beta) \cap B_1(0)$ have a representation of the form $\sum_{i=1}^{+\infty} a_i \beta^{-i}$ (weak-greedy property)

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Moreover, for (complex) Salem bases, all $x \in \mathbb{Q}(\beta) \cap B_1(0)$ have a representation of the form $\sum_{i=1}^{+\infty} a_i \beta^{-i}$ (weak-greedy property)

We cannot get weak-greedy property for non Pisot and non Salem bases (and their complex analogy): start with x big in some embedding where $|\beta|_{\mathfrak{p}} > 1$

Question #2

Given β and $\mathcal{A}$, can we tell whether $\mathbb{Q}(\beta)$ admits eventually periodic $(\beta, \mathcal{A})$-representations?

Theorem (V.)

Let β have no conjugate on the unit circle. The following statements are equivalent

1. $\mathbb{Q}(\beta)$ *admits eventually periodic* $(\beta, \mathcal{A})$*-representations*
2. $\mathbb{Z}[\beta]$ *admits eventually periodic* $(\beta, \mathcal{A})$*-representations*
3. *The set*

   $$X^{\mathcal{A}}(\beta) = \{a_{-k}\beta^k + \cdots + a_{-1}\beta + a_0 : a_i \in \mathcal{A}, k \in \mathbb{N}\}$$

   is relatively dense in $\mathbb{K}_\beta$
4. $0 \in \mathrm{int}(K) \subset \mathbb{K}_\beta$, *where* $\beta K = \bigcup_{a \in \mathcal{A}} (K + a)$