Topology of a class of self-affine tiles in $\mathbb{R}^3$

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Part 1. Self-affine lattice tiles

Self-affine lattice tile

$(\mathbb{Z}^n, M, \mathcal{D})$

- $\mathbb{Z}^n = \{(x_1, x_2, \ldots, x_n)\,;\, x_i \in \mathbb{Z}\}$: subset of $\mathbb{R}^n$.
- An $n \times n$ integer matrix M is expanding, if each eigenvalue is strictly greater than one in modulus.
- Digit set $\mathcal{D} \subset \mathbb{Z}^n$: a complete set of residue class representatives of $\mathbb{Z}^n / M\mathbb{Z}^n$
- The *self-affine tile* $T = T(M, \mathcal{D})$: the unique nonempty compact set satisfying

$$MT = T + \mathcal{D},$$

and $\exists$ **R** $\subset \mathbb{R}^n$ (a discrete set) s.t.

$$\mathbb{R}^n = T + \mathbf{R}, \quad (T^\circ + d) \cap (T^\circ + d') = \emptyset, \quad d, d' \in \mathbf{R}.$$

If in particular $\mathbf{R} = \mathbb{Z}^n$, then we call it self-affine lattice tile or $\mathbb{Z}^n$-tile.

An example

$M = \begin{pmatrix} 0 & -3 \\ 1 & -1 \end{pmatrix}$, $\mathcal{D} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\}$, $T = T(M, \mathcal{D})$ is the $\mathbb{Z}^2$-tile.

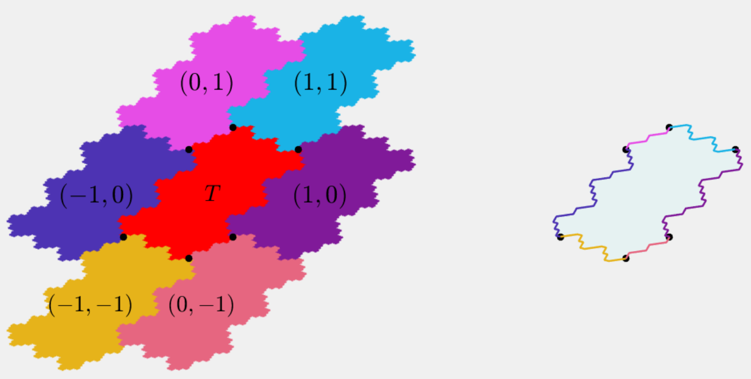

FIGURE – Left : T and translations of T. Right : The boundary of T is a simple closed curve.

Part 2. Neighbour graph

Neighbours

Let $T = T(M, \mathcal{D})$ be an $\mathbb{Z}^n-$tile.

- The set of *neighbours* of T by
$$\mathcal{S} = \{\gamma \in \mathbb{Z}^n \setminus \{0\};\ \ T \cap (T+\gamma) \neq \emptyset\}.$$
  - It is clear that $\mathcal{S}$ is a finite set since T is a compact set.
- Set $B_\gamma = T \cap (T+\gamma)$ for $\gamma \in \mathbb{Z}^n$. If $\gamma \in \mathcal{S}$, we subdivide T, then we obtain
$$\begin{aligned} B_\gamma &= T \cap (T+\gamma) \\ &= M^{-1}(T+\mathcal{D}) \cap M^{-1}(T+\mathcal{D}+M\gamma) \\ &= M^{-1} \bigcup_{d,d' \in \mathcal{D}} (B_{M\gamma+d'-d} + d). \end{aligned}$$
- The *boundary* of T has the following form by non-overlapping property
$$\partial T = \bigcup_{\gamma \in \mathcal{S}} B_\gamma.$$

Neighbour graph : I

Let $T = T(M, \mathcal{D})$ be an $\mathbb{Z}^n-$tile with expanding matrix M and digit set $\mathcal{D}$, $\mathcal{S}$ be the neighbour set of T.

Definition

For a subset $\Gamma \subset \mathbb{Z}^n$ we define a labelled directed graph $G(\Gamma)$ with respect to M and $\mathcal{D}$ as follows.

- Vertex set : elements of Γ ;
- Edge set $E(G(\Gamma))$: there is a labelled edge

  $\gamma \xrightarrow{d|d'} \gamma'$ iff $M\gamma + d' - d = \gamma'$ with $\gamma, \gamma' \in \Gamma$ and $d, d' \in \mathcal{D}$.

We call the directed graph $G(\mathcal{S})$ *neighbor graph* of T. Then ∂T is a graph-directed system directed by the graph $G(\mathcal{S})$. For $\gamma \in \mathcal{S}$,

$$B_\gamma = \bigcup_{d \in \mathcal{D}, \gamma' \in \mathcal{S}, \gamma \xrightarrow{d} \gamma'} M^{-1}(B'_\gamma + d).$$

- For disjoint $\gamma_1, \gamma_2 \in \mathcal{S}$, we set
$$B_{\gamma_1,\gamma_2} = T \cap (T+\gamma_1) \cap (T+\gamma_2).$$
The set of 3-*fold intersection* is then defined by
$$V_2 = \bigcup_{\gamma_1,\gamma_2\in\mathcal{S}} B_{\gamma_1,\gamma_2}.$$
- If B_{γ_1,γ_2} is not empty, we subdivide it.

$$\begin{aligned} B_{\gamma_1,\gamma_2} &= M^{-1}\Big((T+\mathcal{D}) \cap (T+\mathcal{D}+M\gamma_1) \cap (T+\mathcal{D}+M\gamma_2)\Big) \\ &= M^{-1}\Big(\bigcup_{\substack{d,d_1,d_2\in\mathcal{D}\\ \gamma_i'=M\gamma_i+d_i-d\\ i=1,2}} (B_{\gamma_1',\gamma_2'}+d)\Big) \\ &= M^{-1}\Big(\bigcup_{d\in\mathcal{D}}\bigcup_{d_1,d_2\in\mathcal{D}}\bigcup_{\substack{\gamma_i \xrightarrow{d|d_i} \gamma_i'\\ i=1,2}} (B_{\gamma_1',\gamma_2'}+d)\Big). \end{aligned}$$

Neighbour graph II

Definition

Let G be a subgraph of $G(\mathbb{Z}^n)$. The $(\ell+1)$-fold power graph $G_\ell := \times_{j=1}^{\ell} G$ is defined as the reduction $\mathrm{Red}(G'_\ell)$ of the following graph G'_ℓ :

- The states of G'_ℓ are the sets $\{\gamma_1, \dots, \gamma_\ell\}$ consisting of pairwise different states γ_i of G.
- There exists an edge $\{\gamma_{11}, \dots, \gamma_{1\ell}\} \xrightarrow{d} \{\gamma_{21}, \dots, \gamma_{2\ell}\}$ in G'_ℓ if there exist the edges $\gamma_{1i} \xrightarrow{d|d_i} \gamma_{2i} \quad (1 \le i \le \ell)$ in G for certain $d_1, \dots, d_\ell \in \mathcal{D}$.

Then the 3-fold intersection can be written as follows.

$$MB_{\gamma_1,\gamma_2} = \bigcup_{d\in\mathcal{D}} \bigcup_{\{\gamma_1,\gamma_2\}\xrightarrow{d}\{\gamma'_1,\gamma'_2\}\in\times_{j=1}^{2}G(\mathcal{S})} (B_{\gamma'_1,\gamma'_2} + d).$$

Neighbour graph : III

We can characterize $(\ell+1)$-fold intersection with the help of the above graph as follows.

Characterization (Akiyama, Thuswaldner, 2005)

Let $\ell \geq 1$ and choose $\gamma_{01}, \dots, \gamma_{0\ell}$ being pairwise different. Then the following three assertions are equivalent.

(1) $x = \sum_{j\geq 1} M^{-j} d_j \in B_{\gamma_{01},\dots,\gamma_{0\ell}}$.

(2) *There exists an infinite walk of the chain* $\{\gamma_{01}, \dots, \gamma_{0\ell}\} \xrightarrow{d_1} \{\gamma_{11}, \dots, \gamma_{1\ell}\} \xrightarrow{d_2} \{\gamma_{21}, \dots, \gamma_{2\ell}\} \xrightarrow{d_3} \dots$ *in* $\times_{r=1}^{\ell} G(\mathcal{S})$.

(3) *There exists ℓ infinite walks* $\gamma_{0i} \xrightarrow{d_1} \gamma_{1i} \xrightarrow{d_2} \gamma_{2i} \xrightarrow{d_3} \dots \quad (1 \leq i \leq \ell)$ *in* $G(\mathcal{S})$.

For $\ell = 1$, it is exact for neighbour graph. It means for each $\gamma \in \mathcal{S}$, there is at least one infinite walk in $G(\mathcal{S})$ starting from the state γ. This will provide a method to construct the neighbor graph.

Part 3. Our result

Let $1 \leq A \leq B \leq C$ in $\mathbb{Z}$, and $T = T(M, \mathcal{D})$ is the Z^3-tile given by

- Expanding matrix :

$$M = \begin{pmatrix} 0 & 0 & -C \\ 1 & 0 & -B \\ 0 & 1 & -A \end{pmatrix}.$$

- Digit set :

$$\mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} C-1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

We call the $\mathbb{Z}^3$-tile T the ABC-tile.

Let $1 \leq A \leq B \leq C$ in $\mathbb{Z}$, and $T = T(M, \mathcal{D})$ is the Z^3-tile given by

- Expanding matrix :

$$M = \begin{pmatrix} 0 & 0 & -C \\ 1 & 0 & -B \\ 0 & 1 & -A \end{pmatrix}.$$

- Digit set :

$$\mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} C-1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

We ask :

how will be the neighbour graph

$G(S)$ and boundary of the tile ?

Theorem

Let T be an ABC-tile. If A, B, C satisfy $1 \leq A < B < C$ and one of the following conditions

(1) $B \geq 2A - 1$ *and* $C \geq 2(B - A) + 2$ *;*

(2) $B < 2A - 1$ *and* $C \geq A + B - 2$,

then we have the following claims.

- *The neighbour set* $\mathcal{S}$ *has* 14 *elements. In particular, if* $A = B \geq 1$, *it has* 20 *neighbours for all* $C \geq A + B$.
- 3*-fold power* $G_2(\mathcal{S}) = \times_{j=1}^{2} G(\mathcal{S})$ *has* 36 *vertices.*
- 4*-fold power* $G_3(\mathcal{S}) = \times_{j=1}^{3} G(\mathcal{S})$ *has* 24 *vertices. Moreover, this graph contains* 6 *independent circles.*

Neighbours of ABC-tile

We will not go further into the proof and we recommend the paper of Scheicher and Thuswaldner [2002, *Neighbours of self-affine tiles in lattice tilings*] for the calculations of neighbour set.

The 14 neighbours for $1 \leq A < B < C$ are

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} B \\ A \\ 1 \end{pmatrix}, \begin{pmatrix} B-1 \\ A \\ 1 \end{pmatrix}, \begin{pmatrix} A \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} A-1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} B-A \\ A-1 \\ 1 \end{pmatrix}, \begin{pmatrix} B-A+1 \\ A-1 \\ 1 \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -B \\ -A \\ -1 \end{pmatrix}, \begin{pmatrix} 1-B \\ -A \\ -1 \end{pmatrix}, \begin{pmatrix} -A \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1-A \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} A-B \\ 1-A \\ -1 \end{pmatrix}, \begin{pmatrix} A-B-1 \\ 1-A \\ -1 \end{pmatrix} \right\},$$

An example : T is the ABC-tile for $A = 1, B = 2,$ and $C = 4$.
Here following is the neighbour graph of T.

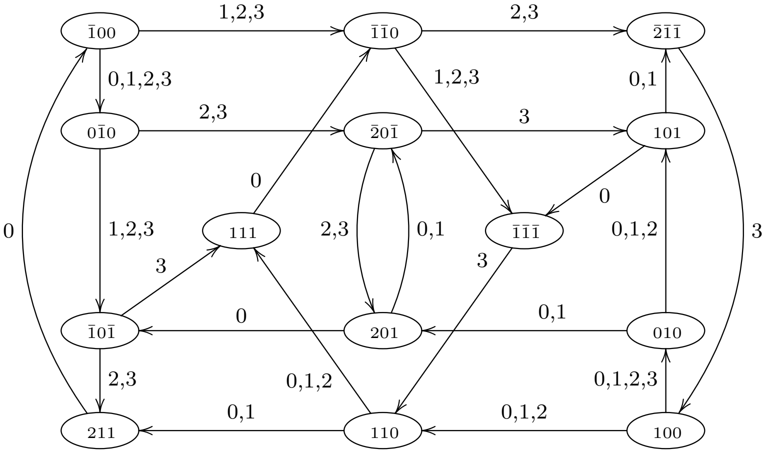

FIGURE – The neighbour graph $G(\mathcal{S})$. The triple abc stands for the vertex $\{(a,b,c)^t\}$ and $\bar{a} = -a$. abc corresponds to the nonempty 2-fold intersection $T \cap (T + (a,b,c)^t)$.

Since we now have all information about the neighbour graph and 2-, 3-fold graphs, we get the topological properties by checking the conditions which were proposed by R. H. Bing (*A characterization of 3-space by partitionings*, 1965).

Theorem

If $1 \leq A < B < C$ *satisfy one of the following conditions*

(1) $B \geq 2A - 1$ *and* $C \geq 2(B - A) + 2$ *;*

(2) $B < 2A - 1$ *and* $C \geq A + B - 2$*,*

then

(1) *The boundary* ∂T *is homeomorphic to a* $2-$*sphere.*

(2) *For every* $\gamma \in \mathcal{S}$*,* ∂B_γ *is a simple closed curve. Moreover, for each disjoint pair* $\gamma, \omega \in \mathcal{S}$*,* $B_{\gamma,\omega}$ *is homeomorphic to a simple curve if* $\{\gamma, \omega\}$ *is a vertex of graph* $G_2(\mathcal{S})$ *and* $B_{\gamma,\omega} = \emptyset$ *otherwise.*

(3) *For each pairwise disjoint* $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{S}$*,* $B_{\gamma_1,\gamma_2,\gamma_3}$ *is a single point if* $\{\gamma_1, \gamma_2, \gamma_3\}$ *is a vertex of directed graph* $G_3(\mathcal{S})$ *and* $B_{\gamma_1,\gamma_2,\gamma_3} = \emptyset$ *otherwise.*

Thank you for your attention.